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A Merton Model of Credit Risk with Jumps

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Abstract: In this note we consider a Merton model for default risk, where the firm's value is driven by a Brownian motion and a compound Poisson process.

Keywords: Merton model, default risk, default probability, processes with jumps

1 Introduction

Various models of Merton's type for credit risk have been studied so far (refer [1] to [8]). This paper aims to our recent results, where a model driven by a jump process is studied in [9] and another model governed by a jumps-diffusion is investigated in [10]. Suppose that the asset value V_t of a company, under a risk neutral measure, is given by the following differential equation

$$dV_t = (r - \beta\lambda)V_t dt + \sigma V_t dW_t + V_{t-} dQ_t, \quad (1.1)$$

where W_t is a standard Brownian motion, $Q(t) = \sum_{i=1}^{N(t)} Y_i$ is a compound Poisson process, $N(t)$ is a Poisson process with intensity $\lambda > 0$, Y_i 's are independent and identically distributed random variables with $E(Y_i) = \beta$. All of these processes are supposed to be considered under the risk neutral measure. In (1.1), r is the interest rate, $\sigma > 0$ is a constant and N_t expresses the number of jumps of Q_t while Y_i is the i -th jump size of $Q(t)$.

The model (1.1) reflects a fact that, the firm's value can change randomly not only in a continuous way but also in a cumulatively discrete fashion.

We will study on the probability of default of the company when its value V_t is less than some debts.

2 Case of one debt L

A bankruptcy situation will occur at some time t when the company asset value is less than a debt L . And the problem is how to calculate the default probability $P(V_t < L)$.

It is known that the solution of (1.1) is given by (see [7])

$$V_t = V_0 \exp\left[\sigma W_t + \left(r - \beta\lambda - \frac{\sigma^2}{2}\right)t\right] \prod_{i=1}^{N_t} (Y_i + 1). \quad (2.1)$$

We see that

$$\ln V_t = \ln V_0 + \sigma W_t + \left(r - \beta\lambda - \frac{\sigma^2}{2}\right)t + \sum_{i=1}^{N_t} \ln(Y_i + 1).$$

And the event $\{V_t < L\}$ or $\{\ln V_t < \ln L\}$ means that

$$\sigma W_t + Z_t < x_t, \quad (2.2)$$

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where

$$x_t = \ln L - (r - \beta\lambda - \frac{\sigma^2}{2})t - \ln V_0, \quad (2.3)$$

$$Z_t = \sum_{i=1}^{N_t} U_i, \quad \text{with } U_i = \ln(1 + Y_i). \quad (2.4)$$

Z_t is also a compound Poisson process where U_i 's are i.i.d. random variables.

We calculate first the characteristic function $\Psi_{Z_t}(s)$ of Z_t . Z_t is also a compound Poisson process where U_i are i.i.d. random variables.

We recall first the characteristic function $\Psi_{Z_t}(s)$ of Z_t :

$$\begin{aligned} \Psi_{Z_t}(s) &= E(e^{isZ_t}) \\ &= \sum_{j=0}^{\infty} E(e^{isZ_t} | N_t = j) P(N_t = j) \\ &= \sum_{j=0}^{\infty} E(e^{is(U_1 + \dots + U_j)}) P(N_t = j) \\ &= \sum_{j=0}^{\infty} (E e^{isU_1} \dots E e^{isU_j}) P(N_t = j) \\ &= \sum_{j=0}^{\infty} (\psi_U(s))^j \frac{(\lambda t)^j}{j!} e^{-\lambda t} = \exp[\lambda t(\psi_U(s) - 1)] \end{aligned} \quad (2.5)$$

where $\psi_U(s)$ is the common characteristic function of U_i 's.

It is known also that, for a compound Poisson process as Z_t we have $\mu(t) = EZ_t = \lambda t E(U_i) = \lambda t E \ln(1 + Y_i) = \lambda t m$; $\sigma^2(t) = \text{Var} Z_t = \lambda t E(U_i^2) = \lambda t E[\ln(1 + Y_i)]^2 = \lambda t \gamma^2$, where $E \ln(1 + Y_i) = m$ and $E[(\ln(1 + Y_i))^2] = \gamma^2$.

Denote by \bar{Z}_t the normalization of Z_t

$$\bar{Z}_t = \frac{Z_t - \mu(t)}{\sigma(t)}.$$

And we will show that Z_t has an approximately normal distribution.

Indeed, according to the Taylor expansion for characteristic function

$$\psi_U(s) = \sum_{k=0}^{\infty} \frac{(is)^k}{k!} E|U|^k,$$

we can write

$$\psi_U(s) = 1 + ism - \frac{\gamma^2}{2}s^2 + o(s^2). \quad (2.6)$$

Now we compute the characteristic function of $\bar{Z}_t = \frac{1}{\sigma(t)}Z_t - \frac{\mu(t)}{\sigma(t)}$,

$$\Psi_{\bar{Z}_t}(s) = e^{-is\frac{\mu(t)}{\sigma(t)}} \Psi_{Z_t}(s/\sigma(t)).$$

Taking account of (2.5) and (2.6) we have

$$\begin{aligned} \Psi_{\bar{Z}_t}(s) &= e^{-is\frac{\mu(t)}{\sigma(t)}} \exp[\lambda t(\psi_U(s/\sigma(t)) - 1)] \\ &= e^{-is\frac{\mu(t)}{\sigma(t)}} \exp[i\lambda t m \frac{s}{\sigma(t)} - \frac{\lambda t \gamma^2}{\sigma(t)^2} \frac{s^2}{2} + o(\frac{s^2}{t})] \\ &= e^{-is\frac{\mu(t)}{\sigma(t)}} \exp[is\mu(t)/\sigma(t) - \frac{\sigma^2(t)}{\sigma^2(t)} \frac{s^2}{2} + o(\frac{s^2}{t})] \\ &= \exp(\frac{-s^2}{2} + o(\frac{s^2}{t})), \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Then $\bar{Z}_t \simeq \mathcal{N}(0, 1)$ or $Z_t \simeq \mathcal{N}(\mu(t), \sigma(t)^2)$, where $\mu(t) = \lambda t E \ln(1 + Y_i)$, $\sigma(t) = \sqrt{\lambda t E [\ln(1 + Y_i)]^2} = \sqrt{\lambda t} \gamma$. Now we can consider $\sigma W_t + Z_t$ as a sum of two independent normal random variables for each t large enough, so it has also a normal distribution with mean

$$\mu^*(t) = \mu(t) = \lambda t E \ln(1 + Y_i)$$

and variance

$$\sigma^*(t) = \sigma^2 + \sigma^2(t) = \sigma^2 + \lambda t E [\ln(1 + Y_i)]^2,$$

where $\sigma > 0$ is a known constant as in (1.1).

And

$$P(\sigma W_t + Z_t < x_t) \approx \Phi\left(\frac{x_t - \mu^*(t)}{\sigma^*(t)}\right), \tag{2.7}$$

where $\Phi(x)$ is the standard normal distribution function.

We are now in the position to state the follow theorem.

Theorem 2.1 *The default probability can be approximated by*

$$P_{default} \approx \frac{1}{\sigma^*(t)\sqrt{2\pi}} \int_{-\infty}^{x_t} e^{-(u - \mu^*(t))^2 / 2\sigma^{*2}(t)} du, \tag{2.8}$$

where

$$\begin{aligned} x_t &= \ln L - (r - \beta\lambda - \sigma^2/2)t - \ln V_0 \\ \mu^*(t) &= \lambda t E \ln(1 + Y_i), \quad \sigma^*(t) = \lambda t E [\ln(1 + Y_i)]^2. \end{aligned}$$

3 Case of many liabilities L_1, L_2, \dots, L_m

Now we consider the case where the company faces up numerous debts L_1, L_2, \dots, L_m that should be paid at times t_1, t_2, \dots, t_m respectively, with $t_1 < t_2 < \dots < t_m = T$.

The company will jump into default position before the time T if and only if at one of time t_i ($i = 1, 2, \dots, m$), it happens that

$$V_{t_i} < L_i.$$

So the probability of default before T is

$$P_{default}(0, T) = 1 - P(V_{t_i} > L_i, \forall t_i).$$

Denote $L = \max\{L_1, \dots, L_m\}$ It is easy to see that for all t_i ($i = 1, \dots, m$) we have

$$(V_{t_i} > L_i) \supset (V_{t_i} > L).$$

Then

$$P_{default}(0, T) \leq 1 - P(V_{t_i} > L, \forall t_i). \tag{3.1}$$

Put $X_t = \sigma W_t + Z_t$, where, as before $Z_t = \sum_{i=1}^{N_t} U_i$, $U_i = \ln(1 + Y_i)$. The inequality $V_{t_i} > L$ is equivalent to

$$X_{t_i} = \sigma W_{t_i} + Z_{t_i} > \ln L - \ln V_0 - (r - \beta\lambda - \frac{\sigma^2}{2})t_i := x_{t_i}.$$

Consider the event

$$A = \{V_{t_i} > L, \forall t_i\} = \bigcap_{i=1}^m \{X_{t_i} > x_{t_i}\}. \tag{3.2}$$

Then

$$P_{default}(0, T) \leq 1 - P(A).$$

It is known that a compound Poisson process is a process of independent increments. The processes (W_t) and (Z_t) are independent and both are of independent increments, so is the process $X_t = \sigma W_t + Z_t$.

Denoting by A_i the event $\{X_{t_i} > x_{t_i}\}$, $i = 1, 2, \dots, m$ we can see that

$$A_1 = \{X_{t_1} > x_{t_1}\} = \{X_{t_1} - X_0 > x_{t_1}\},$$

$$A_2 = \{X_{t_2} > x_{t_2}\} = \{X_{t_2} - X_{t_1} > x_{t_2} - X_{t_1}\} \supset \{X_{t_2} - X_{t_1} > x_{t_2} - x_{t_1}\},$$

if A_1 occurs.

...

$$A_m = \{X_{t_m} > x_{t_m}\} = \{X_{t_m} - X_{t_{m-1}} > x_{t_m} - X_{t_{m-1}}\} \supset \{X_{t_m} - X_{t_{m-1}} > x_{t_m} - x_{t_{m-1}}\},$$

if A_1, \dots, A_{m-1} occur.

Put $B_i = \{X_{t_i} - X_{t_{i-1}} > x_{t_i} - x_{t_{i-1}}\}$ for $i = 1, 2, \dots, m$ and $x_0 = 0$ by convention. It follows that

$$\bigcap_{i=1}^m B_i \subset \bigcap_{i=1}^m A_i = A.$$

Because of the independence of increments we have

$$P(A) \geq P\left(\bigcap_{i=1}^m B_i\right) = \prod_{i=1}^m P(B_i), \quad (3.3)$$

And by definition of B_i ,

$$\begin{aligned} P(B_i) &= P(X_{t_i} - X_{t_{i-1}} > x_{t_i} - x_{t_{i-1}}) \\ &= P(\sigma(W_{t_i} - W_{t_{i-1}}) + (Z_{t_i} - Z_{t_{i-1}}) > x_{t_i} - x_{t_{i-1}}). \end{aligned} \quad (3.4)$$

Put $\bar{X}_i = X_{t_i} - X_{t_{i-1}}$, $\bar{W}_i = \sigma(W_{t_i} - W_{t_{i-1}})$ and $\bar{Z}_i = Z_{t_i} - Z_{t_{i-1}}$, where Z_t is defined as in (2.4). The random variable \bar{W}_i has normal distribution $\mathcal{N}(0, \sigma^2(t_i - t_{i-1}))$. The random variable $\bar{Z}_i = \sum_{k=N_{t_{i-1}}+1}^{N_{t_i}} U_k$ has the same distribution with that of $\sum_{k=1}^{N_{t_i-t_{i-1}}} U_k$ since U_i 's are i.i.d and N_t is a process of stationary and independent increments. We can see that the distribution of \bar{Z}_i is given by

$$\begin{aligned} F_{\bar{Z}_i}(z) &= P(\bar{Z}_i \leq z) = \sum_{n=0}^{\infty} P(N_{t_i-t_{i-1}} = n) P(\bar{Z}_i \leq z / N_{t_i-t_{i-1}} = n) \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} P(\bar{Z}_i \leq z / N_{t_i-t_{i-1}} = n) \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} P\left(\sum_{k=1}^n U_i \leq z\right) \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} F_U^{*n}(z), \end{aligned} \quad (3.5)$$

where F_U^{*n} is the n fold convolution of common distribution of U_k 's.

Suppose now that U_i 's are continuous random variables, so are Z_i 's and \bar{Z}_i 's. Then the density function of $\bar{X}_i = \bar{W}_i + \bar{Z}_i$ is

$$\begin{aligned} f_{\bar{X}_i}(x) &= f_{\bar{W}_i} * f_{\bar{Z}_i}(x) = \int_{-\infty}^{\infty} f_{\bar{W}_i}(x-z) f_{\bar{Z}_i}(z) dz \\ &= \frac{1}{\sigma \sqrt{2\pi}(t_i - t_{i-1})} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-z)^2}{2\sigma^2(t_i - t_{i-1})}\right] f_{\bar{Z}_i}(z) dz, \end{aligned} \quad (3.6)$$

where $f_{\bar{Z}_i}(z) = \frac{d}{dz} F_{\bar{Z}_i}(z)$ is the density function of \bar{Z}_i .

Now we have

$$P(B_i) = 1 - \int_{-\infty}^{x_{t_i} - x_{t_{i-1}}} f_{\bar{X}_i}(x) dx,$$

where $f_{\bar{X}_i}(x)$ is defined by (3.6).

And so, the following assertion is ready to be stated:

Theorem 3.1 If U_i 's are continuous random variables then the probability of default before T is estimated by

$$P_{default}(0, T) \leq 1 - \prod_{i=1}^m \left(1 - \int_{-\infty}^{x_{t_i} - x_{t_{i-1}}} \left[\frac{1}{\sigma \sqrt{2\pi(t_i - t_{i-1})}} \times \int_{-\infty}^{\infty} \exp \left[-\frac{(x-z)^2}{2\sigma^2(t_i - t_{i-1})} \right] f_{\bar{Z}_i}(z) dz \right] dx \right), \tag{3.8}$$

where

$$x_{t_i} = \ln L - \ln V_0 - (r - \beta \lambda - \frac{\sigma^2}{2}) t_i \tag{3.9}$$

and

$$f_{\bar{Z}_i}(z) = \sum_{n=0}^{\infty} \frac{d}{dz} \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} P(\bar{Z}_i \leq z / N_{t_i - t_{i-1}} = n). \tag{3.10}$$

4 Particular cases of Theorem 3.1

We consider some particular cases for distribution of U_k 's.

4.1. Case of normal random variables

Suppose that $U = U_k \sim \mathcal{N}(0, 1)$ then we have $\sum_{k=1}^n U_k \sim \mathcal{N}(0, n)$ with density function $\frac{1}{\sqrt{2\pi n}} e^{-z^2/2n}$ and the density of \bar{Z}_i is

$$f_{\bar{Z}_i}(z) = \frac{1}{\sqrt{2\pi n}} \sum_{n=1}^{\infty} \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} e^{-z^2/2n} \tag{4.1}$$

From (3.8) and (4.1) we have

$$P_{default}(0, T) \leq 1 - \prod_{i=1}^m \left(1 - \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} (t_i - t_{i-1})^n e^{-\lambda(t_i - t_{i-1})} \frac{1}{2\pi\sigma\sqrt{n(t_i - t_{i-1})}} \times \int_{-\infty}^{x_{t_i} - x_{t_{i-1}}} \int_{-\infty}^{\infty} \exp \left[-\frac{(x-z)^2}{2\sigma^2(t_i - t_{i-1})} - \frac{z^2}{2n} \right] dz dx \right). \tag{4.2}$$

4.2. Case of exponential random variable U_k with parameter $\nu > 0$

We know that if $U_k \sim \exp(\nu)$ then $\sum_{k=1}^n U_k \sim \text{Gamma}(n, \nu)$ with the density function

$$\frac{z^{n-1} e^{-z/\nu}}{\nu^n \Gamma(n)},$$

where Γ is Gamma function. Then

$$f_{\bar{Z}_i}(z) = \sum_{n=1}^{\infty} \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} \frac{z^{n-1} e^{-z/\nu}}{\nu^n \Gamma(n)}.$$

We can see the estimation in (3.8):

$$\begin{aligned} P_{default}(0, T) &\leq 1 - \prod_{i=1}^m \left(1 - \int_{-\infty}^{x_{t_i} - x_{t_{i-1}}} \frac{1}{\sigma \sqrt{2\pi(t_i - t_{i-1})}} \int_0^{\infty} \exp \left[-\frac{(x-z)^2}{2\sigma^2(t_i - t_{i-1})} \right] \times \right. \\ &\quad \left. \times \sum_{n=1}^{\infty} \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i - t_{i-1})} \frac{z^{n-1} e^{-z/\nu}}{\nu^n \Gamma(n)} dz dx \right) \\ &= 1 - \prod_{i=1}^m \left(1 - \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} (t_i - t_{i-1})^n e^{-\lambda(t_i - t_{i-1})} \frac{1}{\sigma \sqrt{2\pi(t_i - t_{i-1})}} \times \right. \\ &\quad \left. \times \int_{-\infty}^{x_{t_i} - x_{t_{i-1}}} \int_0^{\infty} \exp \left[-\frac{(x-z)^2}{2\sigma^2(t_i - t_{i-1})} - \frac{z}{\nu} \right] \frac{z^{n-1}}{\nu^n \Gamma(n)} dz dx \right). \end{aligned} \tag{4.3}$$

5 When $U = U_k$'s are general discrete random variables

In this case we have

$$\begin{aligned}
 P(\bar{Z}_i = z) &= P\left(\sum_{k=1}^{N_{t_i-t_{i-1}}} U_k = z\right) = \sum_{n=1}^{\infty} P(N_{t_i-t_{i-1}} = n) P\left(\sum_{k=1}^{N_{t_i-t_{i-1}}} U_k = z / N_{t_i-t_{i-1}} = n\right) \\
 &= \sum_{n=1}^{\infty} P(N_{t_i-t_{i-1}} = n) P\left(\sum_{k=1}^n U_k = z\right) \\
 &= \sum_{n=1}^{\infty} \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} P\left(\sum_{k=1}^n U_k = z\right).
 \end{aligned} \tag{5.1}$$

Denote by \mathcal{L} the set of all possible values of $\bar{Z}_i \equiv \sum_{k=1}^{N_{t_i-t_{i-1}}} U_k$. So that

$$\begin{aligned}
 P(\bar{X}_i < x) &= P(\sigma \bar{W}_i + \bar{Z}_i < x) = \sum_{z \in \mathcal{L}} P(\sigma \bar{W}_i < x - z) P(\bar{Z}_i = z) \\
 &= \sum_{z \in \mathcal{L}} \sum_{n=1}^{\infty} \int_{-\infty}^{x-z} \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \exp\left[-\frac{u^2}{2\sigma^2(t_i-t_{i-1})}\right] \times \\
 &\quad \times \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} P\left(\sum_{k=1}^n U_k = z\right) du.
 \end{aligned} \tag{5.2}$$

The default probability in this case is estimated by

$$\begin{aligned}
 P_{default}(0, T) &\leq 1 - \prod_{i=1}^m \left(1 - \sum_{z \in \mathcal{L}} \sum_{n=1}^{\infty} \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \int_{-\infty}^{x_i-x_{i-1}} \exp\left[-\frac{(x-z)^2}{2\sigma^2(t_i-t_{i-1})}\right] dx \times \right. \\
 &\quad \left. \times \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} P\left(\sum_{k=1}^n U_k = z\right)\right).
 \end{aligned} \tag{5.3}$$

6 U is Poisson random variable with parameter $\beta > 0$

If $U = U_k \sim \text{Poisson}(\beta)$ then

$$\sum_{k=1}^n U_k \sim \text{Poisson}(n\beta)$$

with mass probability

$$p_z = P\left(\sum_{k=1}^n U_k = z\right) = e^{-n\beta} \frac{(n\beta)^z}{z!}, \quad z = 0, 1, 2, \dots$$

Then

$$\begin{aligned}
 P_{default}(0, T) &\leq 1 - \prod_{i=1}^m \left(1 - \sum_{z=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \times \right. \\
 &\quad \times \int_{-\infty}^{x_i-x_{i-1}} \exp\left[-\frac{x^2}{2\sigma^2(t_i-t_{i-1})} - n\beta\right] dx \frac{\lambda^n (t_i - t_{i-1})^n}{n!} e^{-\lambda(t_i-t_{i-1})} \frac{(n\beta)^z}{z!} \left. \right) \\
 &= 1 - \prod_{i=1}^m \left(1 - \sum_{z=0}^{\infty} \sum_{n=1}^{\infty} \frac{\lambda^n (t_i - t_{i-1})^n}{n!} \frac{(n\beta)^z}{z!} \frac{1}{\sigma \sqrt{2\pi(t_i-t_{i-1})}} \times \right. \\
 &\quad \left. \times \int_{-\infty}^{x_i-x_{i-1}} \exp\left[-\frac{x^2}{2\sigma^2(t_i-t_{i-1})} - n\beta\right] dx \right).
 \end{aligned} \tag{6.1}$$

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References

- [1] T. Berg, From actual to risk-neutral default probabilities: Merton and Beyond, *The Journal of Credit Risk*, **6**(2010), no. 1, 55-86.
 - [2] T. Bielecki, M. Jeanblanc and M. Rutkowski, *Credit Risk Modeling*, Center for Study of Insurance and Finance, Osaka University, 2009.
 - [3] H. Bystrom, Merton for Dummies: A Flexible Way of Modelling Default Risk, *Research Paper Series* **112** (2007), Quantitative Finance Research Centre, University of Technology, Sydney.
 - [4] R. Cont and P. Tankov, *Financial Modelling With Jump Processes*, Chapman and Hall, CRC Press, 2003.
 - [5] K. Giesecke and R. G. Lisa, Forecasting Default in Face of Uncertainty, *The Journal of Derivatives*, Fall (2004), 11-25.
 - [6] M. Roger, Merton Robert C. on putting theory into practice, *CFA Magazine*, July-August (2004), 34-37.
 - [7] S. R. Shreve, *Stochastic Calculus for Finance II*, Springer, 2003.
 - [8] Y. Wang, Structural Credit Risk Modeling: Merton and Beyond, *Risk Managements*, June (2009), 30-33.
 - [9] H. T. Phuong Thao and T. H. Thao, A Note on A Model of Merton Type for Valuing Default Risk, *Applied Mathematical Sciences*, **6**(2012), no. 89-92, 4457-4461.
 - [10] H. T. Phuong Thao, Valuing Default Risk for Assets Value Jumps Processes, *East-West J. of Mathematics*, **15**(2013), no. 2, 101-106.
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