Brazilian Journal of Probability and Statistics
2010, Vol. 24, No. 1, 57–67
DOI: 10.1214/08-BJPS013

An approximate approach to fractional stochastic integration and its applications

Nguyen Tien Dung[a] **and Tran Hung Thao**[b]

[a] *FPT University*
[b] *Vietnam Academy of Science and Technology*

Abstract. The aim of this paper is to introduce an approximation approach to fractional stochastic integration. Based on our obtained result, we find explicit solution of some fractional stochastic differential equations and study the ruin probability in the asset liability management (ALM) model.

1 Introduction

The fractional Brownian motion (fBm) and related problems have been investigated by several authors from different approaches [1–4,6–8]. A fBm with Hurst parameter $H \in (0, 1)$ is a Gaussian process $W^H = \{W_t^H, 0 \le t \le T\}$ with $E[W_t^H] = 0$ and the covariance function $R_H(t, s) = E[W_t^H W_s^H]$ defined as

$$R_H(t, s) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}) \tag{1.1}$$

for all $t, s \in [0, T]$.

In [8] Mandelbrot has given a representation of W^H of the form

$$W_t^H = \frac{1}{\Gamma(1+\alpha)}\left[U_t + \int_0^t (t-s)^\alpha \, dW_s\right], \tag{1.2}$$

where W is a standard Brownian motion, $\alpha = H - \frac{1}{2}$ and $U_t = \int_{-\infty}^0 ((t-s)^\alpha - (-s)^\alpha)\, dW_s$. The process $\{U_t, 0 \le t \le T\}$ is of absolutely continuous trajectories. It is known that the second term of (1.2) is the main part expressing the long memory of W_t^H and is called a fractional Brownian motion of Liouville form [3, 6,9].

In this paper we consider the fBm of Liouville form with parameter $H \in (\frac{1}{2}, 1)$

$$B_t = \int_0^t (t-s)^\alpha \, dW_s. \tag{1.3}$$

In [1] Alòs, Mazet and Nualart used the Mallivin Calculus method to approximate B_t by semimartingales B_t^ε defined as follows:

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha \, dW_s, \qquad \varepsilon > 0. \tag{1.4}$$

Key words and phrases. Fractional Brownian motion, ruin probability, asset liability management.

Received August 2007; accepted July 2008.

And an important result was given by Thao [9] who has proved that $B_t^\varepsilon \overset{L^2(\Omega)}{\longrightarrow} B_t$ when $\varepsilon \to 0^+$ and that the convergence is uniform with respect $t \in [0, T]$. Using Thao's results, we prove that for a stochastic process $f \in \bigcup_{\mu>3/4} C^\mu[0, T]$ we have

$$\int_0^t f(s)\, dB_s^\varepsilon \overset{L^1(\Omega)}{\longrightarrow} \int_0^t f(s)\, d\mathrm{B}_s \qquad \forall t \in [0, T],$$

where the left-hand side is an integral with respect to a semimartingale and the right-hand side is a pathwise integral, which is constructed by Zähle [12,13]. We use the obtained results to study a class of fractional stochastic differential equations and the ruin probability in the asset liability management (ALM) model.

2 Preliminaries

For the sake of convenience, we recall an important result, which will be the basis of this paper. For every $\varepsilon > 0$ we define

$$B_t^\varepsilon = \int_0^t (t - s + \varepsilon)^\alpha \, dW_s, \qquad \alpha = H - \frac{1}{2}. \tag{2.1}$$

We have the following theorem:

Theorem 2.1. *Let $H \in (0, 1)$. Then*

I. *The process $\{B_t^\varepsilon, 0 \le t \le T\}$ is a semimartingale. Moreover*

$$B_t^\varepsilon = \int_0^t \varphi_\varepsilon(s)\, ds + \varepsilon^\alpha W_t, \tag{2.2}$$

where $\varphi_\varepsilon(t) = \alpha \int_0^t (t - u + \varepsilon)^{\alpha-1}\, dW_u$.

II. *The process B_t^ε converges to B_t in $L^2(\Omega)$ when ε tends 0. This convergence is uniform with respect to $t \in [0, T]$.*

Proof. The detailed proof of this theorem can be found in [9]. In case $H > \frac{1}{2}$, we suggest a different way, as follows:

For all $a, b > 0$ we have the following inequality

$$(a + b)^\alpha \le a^\alpha + b^\alpha \qquad \forall \alpha \in [0, 1]. \tag{2.3}$$

Applying this inequality to $a = t - u, b = \varepsilon$ we obtain

$$\begin{aligned} E|B_t^\varepsilon - B_t|^2 &= E\left(\int_0^t [(t - u + \varepsilon)^\alpha - (t - u)^\alpha]\, dW_u\right)^2 \\ &= \int_0^t [(t - u + \varepsilon)^\alpha - (t - u)^\alpha]^2\, du \\ &\le \int_0^t \varepsilon^{2\alpha}\, du = \varepsilon^{2\alpha} t \qquad \forall t \in [0, T]. \end{aligned}$$

So

$$E|B_t^\varepsilon - B_t|^2 \le T\varepsilon^{2\alpha} \qquad \forall t \in [0,T]. \tag{2.4}$$

□

Remark 2.1. We recall some results on a generalization of the Stieltjes integral introduced by Zähle [12].

(i) We have the following estimate for all $t \in [0,T]$ and $f \in W^{\lambda,1}[0,t]$, $g \in W^{1-\lambda,\infty}[0,T]$

$$\left|\int_0^t f\,dg\right| \le C(\lambda)\|f\|_{\lambda,1}\|g\|_{1-\lambda,\infty}. \tag{2.5}$$

(ii) If $f \in C^\lambda[0,T]$ and $g \in C^\mu[0,T]$ with $\lambda + \mu > 1$, it is proved that the integral $\int_0^t f\,dg$ coincides with the Riemann–Stieltjes integral.

3 Main results

Proposition 3.1. *Suppose that $H \in (\frac{1}{2}, 1)$ and $\varepsilon \in (0,1)$.*

(a) *The following estimates hold for all $t, s \in [0,T]$*

$$E|B_t - B_s|^2 \le c_1|t-s|^{2H}, \tag{3.1}$$

$$E|B_t^\varepsilon - B_s^\varepsilon|^2 \le c_1|t-s|, \tag{3.2}$$

where c_1 is a positive constant, depends only on H and T.

(b) *Put $D_t^\varepsilon = B_t^\varepsilon - B_t$ then*

$$E|D_t^\varepsilon - D_s^\varepsilon|^2 \le c_2\varepsilon^\alpha|t-s|^{1/2} \qquad \forall t, s \in [0,T], \tag{3.3}$$

where c_2 is some constant depending only on H and T.

(c) *For all $0 < \lambda < \frac{1}{4}$ we have the following estimate*

$$E\|B^\varepsilon - B\|_{\lambda,1} \le c_3\varepsilon^{\alpha/2},$$

where c_3 depends only on H, T and λ.

Proof. (a) The inequality (3.1) is elementary property of fBm and its proof can be found in [6]. The inequality (3.2) can be proved as follows:

Without loss of generality we may assume that $s \le t$. By virtue of the Itô isometry we see that

$$E|B_t^\varepsilon - B_s^\varepsilon|^2 = \int_0^t (t-u+\varepsilon)^{2\alpha}\,du + \int_0^s (s-u+\varepsilon)^{2\alpha}\,du$$
$$- 2\int_0^s (t-u+\varepsilon)^\alpha(s-u+\varepsilon)^\alpha\,du.$$

We consider the right-hand side of the latest equality as a function in $t \in [s, T]$, denoted by $f(t)$. It is clear that $f \in C^\infty[s, T]$ and we have $f(s) = 0$, $f'(s) = \varepsilon^{2\alpha}$, $f'(t) = (t+\varepsilon)^{2\alpha} - 2\alpha \int_0^s (t-u+\varepsilon)^{\alpha-1}(s-u+\varepsilon)^\alpha\, du$. Therefore

$$\begin{aligned}|f'(t)| &\le (t+\varepsilon)^{2\alpha} + 2\alpha \int_0^s (s-u+\varepsilon)^{2\alpha-1}\, du \\ &= (t+\varepsilon)^{2\alpha} + (s+\varepsilon)^{2\alpha} - \varepsilon^{2\alpha} \le 2(T+1)^{2\alpha} \qquad \forall s \le t \le T.\end{aligned}$$

The theorem of finite increment applied to the function $f(t)$ yields

$$f(t) = |f(t) - f(s)| \le 2(T+1)^{2\alpha}|t-s|.$$

(b) Using (2.4) we obtain

$$E|D_t^\varepsilon - D_s^\varepsilon|^2 \le 2(E|D_t^\varepsilon| + E|D_s^\varepsilon|) \le 4T\varepsilon^{2\alpha}. \tag{3.4}$$

On the other hand,

$$\begin{aligned}E|D_t^\varepsilon - D_s^\varepsilon|^2 &\le 2(E|B_t^\varepsilon - B_s^\varepsilon|^2 + E|B_t - B_s|^2) \\ &\le 2c_1\big(1 + (2T)^{2H-1}\big)|t-s|.\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$E|D_t^\varepsilon - D_s^\varepsilon|^2 \le c_2\varepsilon^\alpha|t-s|^{1/2},$$

where $c_2 = \sqrt{8Tc_1((2T)^{2H-1}+1)}$.

(c) We have

$$E\|B^\varepsilon - B\|_{\lambda,1} = \int_0^T \frac{E|D_s^\varepsilon|}{s^\lambda}\, ds + \int_0^T\int_0^s \frac{E|D_s^\varepsilon - D_y^\varepsilon|}{(s-y)^{\lambda+1}}\, dy\, ds.$$

Using the estimates (2.4) and (3.3) we obtain

$$\begin{aligned}E\|B^\varepsilon - B\|_{\lambda,1} &\le \int_0^T \frac{\varepsilon^\alpha\sqrt{T}}{s^\lambda}\, ds + \int_0^T\int_0^s \frac{\sqrt{c_2\varepsilon^\alpha\sqrt{(s-y)}}}{(s-y)^{\lambda+1}}\, dy\, ds \\ &\le \varepsilon^{\alpha/2}\bigg(\int_0^T \frac{\sqrt{T}}{s^\lambda}\, ds + \int_0^T\int_0^s \frac{\sqrt{c_2}}{(s-y)^{\lambda+3/4}}\, dy\, ds\bigg).\end{aligned}$$

The integrals in the bracket above are finite because of $0 < \lambda < 1/4$.

Thus, there exists a finite constant c_3 such that

$$E\|B^\varepsilon - B\|_{\lambda,1} \le c_3\varepsilon^{\alpha/2}.$$

The proof is complete. □

The next theorem is a basic result of this paper.

Theorem 3.1. *Suppose that a stochastic process $f:[0,T]\times\Omega\to\mathbb{R}$, satisfies the following condition: For some $\delta>0$ there exists a finite constant $M>0$ such that*

$$\|f\|_{C^{3/4+\delta}[0,T]}<M \qquad a.s.$$

Then for all $t\in[0,T]$

$$\int_0^t f(s)\,dB_s^\varepsilon \xrightarrow{L^1(\Omega)} \int_0^t f(s)\,d\mathrm{B}_s \qquad as\ \varepsilon\to 0.$$

Proof. We first recall that

$$E|B_t-B_s|^2\le c_1|t-s|^{2H}$$

for all $t,s\in[0,T]$. As a consequence, the process B has β-Hölder continuous path for all $0<\beta<H$, that is, $B\in\bigcap_{0<\beta<H}C^\beta[0,T]$ with probability one. Moreover, the process $f\in C^{3/4+\delta}[0,T]$. It follows from Remark 2.1 that the integral $\int_0^t f(s)\,d\mathrm{B}_s$ can be understood as a Riemann–Stieltjes integral. An application of the integration-by-parts formula to both integrals $\int_0^t f(s)\,d\mathrm{B}_s$ and $\int_0^t f(s)\,dB_s^\varepsilon$ we obtain

$$\begin{aligned}\int_0^t f(s)\,dB_s^\varepsilon-\int_0^t f(s)\,d\mathrm{B}_s&=\int_0^t f(s)\,d(B_s^\varepsilon-B_s)\\&=f(t)(B_t^\varepsilon-B_t)-\int_0^t(B_s^\varepsilon-B_s)\,df(s).\end{aligned}$$

Hence

$$\left|\int_0^t f(s)\,dB_s^\varepsilon-\int_0^t f(s)\,d\mathrm{B}_s\right|\le|f(t)(B_t^\varepsilon-B_t)|+\left|\int_0^t(B_s^\varepsilon-B_s)\,df(s)\right|.$$

Applying the Hölder inequality and the relation (2.4) yields

$$E|f(t)(B_t^\varepsilon-B_t)|\le\|f(t)\|\|B_t^\varepsilon-B_t\|\le M\sqrt{T}\varepsilon^\alpha, \tag{3.6}$$

where $\|\cdot\|$ stands for the $L^2(\Omega)$-norm.

Moreover, since $f\in C^{3/4+\delta}[0,T]\subset W^{(3+\delta)/4,\infty}[0,T]$ a.s., it follows from the inequality (2.5) with $\lambda=\frac{1-\delta}{4}$ that

$$\left|\int_0^t(B_s^\varepsilon-B_s)\,df(s)\right|\le C(\delta)\|B^\varepsilon-B\|_{(1-\delta)/4,1}\|f\|_{(3+\delta)/4,\infty}.$$

An easy computation leads us to the inequality

$$\|f\|_{(3+\delta)/4,\infty}\le M\left(1+\frac{4}{3\delta}\right)T^{3\delta/4}:=M_1 \qquad \text{a.s.}$$

Hence from Proposition 3.1 we see that

$$E\left|\int_0^t(B_s^\varepsilon-B_s)\,df(s)\right|\le M_1C(\delta)c_3\varepsilon^{\alpha/2}.$$

Combining this inequality and (3.6) we have

$$E\left|\int_0^t f(s)\,dB_s^\varepsilon - \int_0^t f(s)\,d\mathrm{B}_s\right| < c_4\varepsilon^{\alpha/2}$$

with $c_4 = M\sqrt{T} + M_1C(\delta)c_3$.

The theorem is thus proved. □

4 An application to fractional stochastic differential equation

In this section, we will give the explicit solution for an important class of fractional stochastic differential equations of the form

$$\begin{cases} dX_t = b(t, X_t)\,dt + \sigma\,dB_t, \\ X_t|_{t=0} = X_0, \end{cases} \tag{4.1}$$

where $b(t,x) = \alpha(t)x + \beta(t)$ and σ is a constant. We assume that $\alpha(t), \beta(t)$ are deterministic functions on $[0, T]$ and $\alpha(t)$ is a function bounded by some constant $K > 0$.

A classical example of equations (4.1) is the Langevin equation

$$dX_t = (\alpha - bX_t)\,dt + \sigma\,dB_t.$$

We consider the corresponding approximation equation

$$\begin{cases} dX_t^\varepsilon = b(t, X_t^\varepsilon)\,dt + \sigma\,dB_t^\varepsilon, \\ X_t^\varepsilon|_{t=0} = X_0. \end{cases} \tag{4.2}$$

By extending a result by Thao and Nguyen [11] we will prove that the solution of the equation (4.1) is the limit in $L^1(\Omega)$ of the solution of (4.2) as ε tends to 0.

Proposition 4.1. *Suppose that $H \in (\frac{1}{2}, 1)$. Then the solution X_t^ε of the equation (4.2) converges to the solution X_t of (4.1) in $L^1(\Omega)$ as $\varepsilon \to 0$ uniformly with respect to $t \in [0, T]$.*

Proof. We have

$$X_t^\varepsilon = X_0^\varepsilon + \int_0^t b(s, X_s^\varepsilon)\,ds + \sigma\int_0^t dB_s^\varepsilon,$$

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \sigma\int_0^t dB_s$$

then

$$\begin{aligned} |X_t^\varepsilon - X_t| &\le \int_0^t |b(s, X_s^\varepsilon) - b(s, X_s)|\,ds + \sigma|B_t^\varepsilon - B_t| \\ &\le \int_0^t |\alpha(s)(X_s^\varepsilon - X_s)|\,ds + \sigma|B_t^\varepsilon - B_t|. \end{aligned}$$

Using the estimate (2.4) and since $|\alpha(s)| \le K \ \forall s \in [0, T]$ we obtain

$$E|X_t^\varepsilon - X_t| \le K \int_0^t E|X_s^\varepsilon - X_s|\, ds + \sigma\sqrt{T}\varepsilon^\alpha \tag{4.3}$$

for all $0 \le t \le T$. A standard application of Gronwall's lemma starting from (4.3) will give us

$$E|X_t^\varepsilon - X_t| \le \sigma\sqrt{T}\varepsilon^\alpha e^{Kt}.$$

It follows that

$$\sup_{0\le t\le T} E|X_t^\varepsilon - X_t| \le \sigma\sqrt{T}\varepsilon^\alpha e^{KT}.$$

So $X_t^\varepsilon \xrightarrow{L^1(\Omega)} X_t$ when $\varepsilon \to 0^+$, and the convergence is uniform with respect to $t \in [0, T]$. □

Next, we will find the explicit solution of the equation (4.2).

Proposition 4.2. *Suppose that $H \in (\frac{1}{2}, 1)$ and X_0 is a random variable. Then the solution of (4.2) is given by*

$$X_t^\varepsilon = e^{\int_0^t \alpha(u)\,du}\left(X_0 + \int_0^t \beta(s)e^{-\int_0^s \alpha(u)\,du}\, ds + \sigma \int_0^t e^{-\int_0^s \alpha(u)\,du}\, dB_s^\varepsilon\right).$$

Proof. By (2.2) we can rewrite the equation (4.2) in the following form:

$$dX_t^\varepsilon = \big(\alpha(t)X_t^\varepsilon + \beta(t) + \sigma\varphi_\varepsilon(t)\big)\, dt + \sigma\varepsilon^\alpha\, dW_t. \tag{4.4}$$

We split (4.4) into two equations:

$$dX_1(t) = \big(\alpha(t)X_1(t) + \beta(t)\big)\, dt + \sigma\varepsilon^\alpha\, dW_t, \tag{4.5}$$

$$dX_2(t) = \big(\alpha(t)X_2(t) + \sigma\varphi_\varepsilon(t)\big)\, dt. \tag{4.6}$$

The solution of (4.2) will be $X_t^\varepsilon = X_1(t) + X_2(t)$. We see that (4.5) is an Itô stochastic differential equation and its solution is given by

$$X_1(t) = e^{\int_0^t \alpha(u)\,du}\left(X_1(0) + \int_0^t \beta(s)e^{-\int_0^s \alpha(u)\,du}\, ds + \sigma\varepsilon^\alpha \int_0^t e^{-\int_0^s \alpha(u)\,du}\, dW_s\right).$$

The equation (4.6) is an ordinary differential equation for every fixed ω and its solution is

$$X_2(t) = e^{\int_0^t \alpha(u)\,du}\left(X_2(0) + \sigma \int_0^t \varphi_\varepsilon(s)e^{-\int_0^s \alpha(u)\,du}\, ds\right).$$

Thus, noting that $dB_s^\varepsilon = \varphi_\varepsilon(s)\, ds + \varepsilon^\alpha\, dW_s$, we see that the solution of (4.2) is

$$\begin{aligned} X_t^\varepsilon &= X_1(t) + X_2(t) \\ &= e^{\int_0^t \alpha(u)\,du}\left(X_0 + \int_0^t \beta(s)e^{-\int_0^s \alpha(u)\,du}\, ds + \sigma \int_0^t e^{-\int_0^s \alpha(u)\,du}\, dB_s^\varepsilon\right). \end{aligned}$$

The proposition is proved. □

Now the most important result of this section can be stated as follows.

Theorem 4.1. *Suppose that $H \in (\frac{1}{2}, 1)$ and X_0 is a random variable such that $E|X_0| < \infty$. Then the solution of (4.1) is the unique and given by*

$$\mathrm{X}_t = e^{\int_0^t \alpha(u)\,du}\left(X_0 + \int_0^t \beta(s) e^{-\int_0^s \alpha(u)\,du}\,ds + \sigma \int_0^t e^{-\int_0^s \alpha(u)\,du}\,d\mathrm{B}_s\right).$$

Proof. First of all, by Proposition 4.1 and Proposition 4.2 we have only to prove that

$$\int_0^t f(s)\,dB_s^{\varepsilon} \xrightarrow{L^1(\Omega)} \int_0^t f(s)\,d\mathrm{B}_s \qquad \forall t \in [0, T], \tag{4.7}$$

where $f(s) = \exp(-\int_0^s \alpha(u)\,du)$. This is obvious since $f \in C^1[0, T]$.

The uniqueness of the solution of (4.1) follows from that of L^1-limit. If $X_t^{(1)}$ and $X_t^{(2)}$ are limits of X_t^{ε} in $L^1(\Omega)$, then

$$E|X_t^{(1)} - X_t^{(2)}| \le E|X_t^{(1)} - X_t^{\varepsilon}| + E|X_t^{(2)} - X_t^{\varepsilon}| \to 0 \qquad \text{as } \varepsilon \to 0.$$

This complete the proof. □

5 The ruin probability in the ALM model

In finance and economics, it is usual to model the evaluation of the assets and the liabilities of a bank or an insurance company with the use of stochastic processes for both parts of the balance sheet. This leads to useful models used in theory and practice of the asset liability management (ALM).

In this section, we consider an ALM model where the asset X_t and the liability Y_t satisfy the following stochastic differential equations:

$$\begin{cases} dX_t = \mu_1 X_t\,dt + \sigma_1 X_t\,dB_t^{(1)}, \\ dY_t = \mu_2 Y_t\,dt + \sigma_2 Y_t\,dB_t^{(2)}, \\ X|_{t=0} = X_0, \qquad Y|_{t=0} = Y_0 < X_0, \end{cases} \tag{5.1}$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2$ are non-negative parameters.

$B_t^{(1)} = \int_0^t (t-s)^{\alpha}\,dW_s^{(1)}$, $B_t^{(2)} = \int_0^t (t-s)^{\alpha}\,dW_s^{(2)}$ being two fractional Brownian motions whose correlation coefficient is ρ with $|\rho| \le 1$.

It is known from [9,10] that the solution of fractional Black–Scholes model

$$dX_t = \mu X_t\,dt + \sigma X_t\,dB_t$$

is $S_t = S_0 e^{\mu t + \sigma B_t}$. It follows from this fact that

$$X_t = X_0 e^{\mu_1 t + \sigma_1 B_t^{(1)}}, \qquad Y_t = Y_0 e^{\mu_2 t + \sigma_2 B_t^{(2)}}$$

and

$$\frac{X_t}{Y_t} = \frac{X_0}{Y_0} \exp\bigl((\mu_1 - \mu_2)t + \sigma_1 B_t^{(1)} - \sigma_2 B_t^{(2)}\bigr).$$

Note that $W^{(1)}, W^{(2)}$ have the same ρ for their correlation coefficient ρ because $B^{(1)}, B^{(2)}$ have correlation coefficient ρ. Hence

$$\sigma_2 W_t^{(2)} - \sigma_1 W_t^{(1)}$$

is probabilistically equivalent to the process σW_t, where W_t is a standard Brownian motion and

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \tag{5.2}$$

We obtain

$$\begin{aligned}\sigma_1 B_t^{(1)} - \sigma_2 B_t^{(2)} &= \int_0^t (t-s)^\alpha \, d\bigl(\sigma_1 W_s^{(1)} - \sigma_2 W_s^{(2)}\bigr) \\ &= -\sigma \int_0^t (t-s)^\alpha \, dW_s =: -\sigma B_t\end{aligned}$$

and

$$\frac{X_t}{Y_t} = \frac{X_0}{Y_0} \exp(\mu t - \sigma B_t), \qquad \mu = \mu_1 - \mu_2. \tag{5.3}$$

We now see that the lifetime τ of the bank or the insurance company can be naturally defined as the first value of t such that $X_t < Y_t$, or equivalently

$$\tau = \inf\left\{ t : \ln \frac{X_t}{Y_t} < 0 \right\}$$

and the ruin probability with a finite time horizon $[0, t]$ is

$$\varphi(X_0, Y_0, t) := P(\tau < t) = P\left(\ln \frac{X_s}{Y_s} < 0 \text{ for some } s < t \right)$$

and for infinite time horizon

$$\varphi(X_0, Y_0) := \lim_{t \to \infty} \varphi(X_0, Y_0, t).$$

It follows from (5.3) that

$$\begin{aligned}\varphi(X_0, Y_0) &= P\left(\ln \frac{X_t}{Y_t} < 0 \text{ for some } t \geq 0 \right) \\ &= P(-\mu t + \sigma B_t > u \text{ for some } t \geq 0) \\ &= P\Bigl(\sup_{t \geq 0} (-\mu t + \sigma B_t) > u \Bigr),\end{aligned}$$

where $u = \ln \frac{X_0}{Y_0}$. In order to estimate $\varphi(X_0, Y_0)$ we use a result by Dębicki [5], Corollary 4.1, which says that

Proposition 5.1. *For* $\frac{1}{2} \le H \le 1$

$$\lim_{u\to\infty} \frac{1}{u^{2-2H}} \ln P\big(A(B_H^o, c) > u\big) = -h, \tag{5.4}$$

where $B_H^o(t) = \sqrt{2H} B_t$, $A(B_H^o, c) = \sup\{B_H^o(t) - ct : t \ge 0\}$ *and*

$$h = \frac{1}{2}\left(\frac{c}{H}\right)^{2H}\left(\frac{1}{1-H}\right)^{2-2H}.$$

Now we can state the following theorem:

Theorem 5.1. *If* $\mu_1 \ge \mu_2$ *then the ruin probability for the ALM model* (5.1) *satisfies the relation*:

$$\lim_{u\to\infty} \frac{\ln \varphi(X_0, Y_0)}{u^{2-2H}} = -\frac{\mu^{2H}}{H\sigma^2}\left(\frac{H}{1-H}\right)^{2-2H}, \tag{5.5}$$

where $\mu = \mu_1 - \mu_2, \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ *and* $u = \ln \frac{X_0}{Y_0}$.

Proof. We have

$$\begin{aligned}\varphi(X_0, Y_0) &= P\left(\sup_{t\ge 0}\left(B_t - \frac{\mu}{\sigma}t\right) > \frac{u}{\sigma}\right) \\ &= P\left(\sup_{t\ge 0}\left(B_H^o(t) - \frac{\sqrt{2H}\mu}{\sigma}t\right) > \frac{\sqrt{2H}u}{\sigma}\right)\end{aligned}$$

therefore (5.5) follows from Proposition 5.1. The theorem is completed. □

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments in improving the paper.

References

[1] Alòs, E., Mazet, O. and Nualart, D. (2000). Stochastic calculus with respect to fractional Brownian motion with Hurst paramenter less than $\frac{1}{2}$. *Stochastic Processes and Their Applications* **86** 121–139. MR1741199

[2] Androshchuk, T. O. (2006). Approximation of a stochastic integral with respect to fractional Brownian motion by integrals with respect to absolutely continuous processes. *Theory of Probability and Mathematical Statistics* **73** 19–29. MR2213333

[3] Comte, F. and Renault, E. (1998). Long memory in continuous-time stochastic volatility models. *Mathematical Finance* **8** 291–323. MR1645101

[4] Duncan, T. E., Hu, Y. and Pasik Duncan, B. (2000). Stochastic calculus for fractional Brownian motion. *SIAM Control and Optimization* **38** 582–612. MR1741154

[5] Dębicki, K., Michna, Z. and Rolski, T. (1998). On the supremum from Gaussian processes over infinite horizon. *Probability and Mathematics Statistics* **18** 83–100. MR1644033

[6] Feyel, D. and De la Pradelle, A. (1999). On fractional Brownian processes. *Potential Analysis* **10** 273–288. MR1696137

[7] Hu, Y. and Øksendal, B. (1999). Fractional white noise calculus and applications to finance. Preprint, Dept. Mathematics, Univ. Oslo, 10. MR1976868

[8] Mandelbrot, B. and van Ness, J. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Review* **10** 422–437. MR0242239

[9] Thao, T. H. (2006). An approximate approach to fractional analysis for finance. *Nonlinear Analysis* **7** 124–132. (Available also online on Science Direct.) MR2183802

[10] Thao, T. H. and Thomas-Agnan, C. (2003). Evolution des cours gouvernée par un processus de type ARIMA fractionnaire. *Studia Univ. Babes-Bolyai Math.* **48** 107–115. MR2110323

[11] Thao, T. H. and Nguyen, T. T. (2003). Fractal Langevin equation. *Vietnam Journal of Mathematics* **30** 89–96. MR1933800

[12] Zähle, M. (1998). Integration with respect to fractal functions and stochastic calculus. *Part I. Probability Theory and Related Fields* **111** 333–374. MR1640795

[13] Zähle, M. (1999). On the link between fractional and stochastic calculus. In *Stochastic Dynamics* 305–325. Springer, New York. MR1678495

Department of Mathematics
FPT University
15B Pham Hung Street, Cau Giay Hanoi
Vietnam
E-mail: dungnt@fpt.edu.vn

Institute of Mathematics
Vietnam Academy of Science and Technology
Vietnam
E-mail: ththao@math.ac.vn