Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2015,**35B**(3):610–618
Acta Mathematica Scientia 数学物理学报
http://actams.wipm.ac.cn

# ASYMPTOTIC BEHAVIOR OF LINEAR ADVANCED DIFFERENTIAL EQUATIONS*

*Nguyen Tien DUNG*
*Department of Mathematics, FPT University 8 Ton That Thuyet, My Dinh, Tu Liem, Hanoi, VietNam*
*E-mail*: *dung_nguyentien10@yahoo.com Cc: dungnt@fpt.edu.vn*

**Abstract** In this article, we consider a general class of linear advanced differential equations, and obtain explicitly sufficient conditions of convergence and exponential convergence to zero. A necessary condition is provided as well.

**Key words** Advanced differential equations; asymptotic behaviors; fixed points

**2010 MR Subject Classification** 34K06; 34D05; 47H10

## 1 Introduction

Advanced differential equations were first discussed by Myschkis (1955) [17] and Bellman & Cooke (1963) [2]. Such equations represent a system in which the rate of change of a quantity depends on present and future values of the quantity and were proved to be valuable tools to model the dynamics of many processes in various fields of science and engineering. Indeed, we can find numerous applications in optimal control problems with delay [18], population growth [6, 15], population genetics [1], neural networks [7], the field of time symmetric electrodynamics [20], the study of wavelets [22, 23], and economics [8, 11], etc.

In 1980's, the existence and uniqueness of the solution to linear and nonlinear advanced differential equations were investigated by Shah et al [19, 21], and the oscillation properties of the solution were studied by Kitamura & Kusano [12], and Ladas & Stavroulakis [14]. Since then, particularly in the last decade, the fundamental problems in the theory of advanced differential equations have attracted much attention by different authors; for example, the existence and uniqueness of the solution [10], the oscillation properties [4, 13, 16], and numerical approximations [9]. However, a general theory of advanced differential equations has not yet been developed completely. To our best knowledge, the results regarding stability properties of the solution are scarce. The aim of this article is to partially fill up this gap. We consider the linear advanced differential equations of the form

$$\dot{x}(t)+a(t)x(t+h(t))+b(t)x(t+r(t))=0,\ \ t\geq t_0, \tag{1.1}$$

where $a(t)$ and $b(t)$ are continuous on $[t_0,\infty)$, and the advanced arguments $h(t), r(t)$ are continuous functions with $h(t)\geq 0$ and $r(t)\geq 0$.

*Received January 7, 2014; revised May 6, 2014.

As pointed out by Bellman & Cooke [2], the standard techniques developed for delay differential equations can not be effectively applied to (1.1) even when $a(t), b(t), h(t)$, and $r(t)$ are constants. About fifteen years ago, a new technique of fixed points was developed for studying stability of delay differential equations (see, for example, [5]). In this article, we show that this technique can work on (1.1) and then establish sufficient conditions ensuring that the solution (exponentially) converges to zero. We also obtain a necessary condition for the convergence to zero.

The rest of this article is organized as follows. Section 2 contains the main results of this article. In Section 3, we generalize the results to the advanced differential equations with several terms; an example is also given in this section.

## 2 Main Results

In this section, we state and prove our main results. Before doing these, let us recall a definition of the solution to (1.1).

**Definition 2.1** A continuously differentiable function $x : [t_0, \infty) \to \mathbb{R}$ is called a solution of equation (1.1), if it satisfies the relation (1.1) for all $t \geq t_0$.

We need a technical lemma which plays a key role in this article. This lemma transforms (1.1) into an equivalent integral equation for which the method of fixed points can work.

**Lemma 2.2** Let $x(t)$ be the solution of (1.1) on $[t_0, \infty)$. Then, $x(t)$ satisfies the following integral equation

$$x(t) = x_0 e^{-\int_{t_0}^t D(u)\mathrm{d}u} + \int_{t_0}^t e^{-\int_s^t D(u)\mathrm{d}u} a(s)\Big(\int_s^{s+h(s)} E_x(u)\mathrm{d}u\Big)\mathrm{d}s$$
$$+ \int_{t_0}^t e^{-\int_s^t D(u)\mathrm{d}u} b(s)\Big(\int_s^{s+r(s)} E_x(u)\mathrm{d}u\Big)\mathrm{d}s, \quad t \geq t_0, \tag{2.1}$$

where $x_0 = x(t_0)$, $D(t) = a(t) + b(t)$, and $E_x(t) = a(t)x(t + h(t)) + b(t)x(t + r(t))$.

**Proof** Using the relation

$$x(u) - x(t) = \int_t^u \dot{x}(s)\mathrm{d}s,$$

we can rewrite equation (1.1) as follows,

$$\dot{x}(t) = -[a(t) + b(t)]x(t) - a(t)\int_t^{t+h(t)} \dot{x}(s)\mathrm{d}s - b(t)\int_t^{t+r(t)} \dot{x}(s)\mathrm{d}s.$$

After substituting $\dot{x}$ from (1.1), we obtain

$$\dot{x}(t) = -[a(t) + b(t)]x(t) + a(t)\int_t^{t+h(t)} \Big(a(s)x(s + h(s)) + b(s)x(s + r(s))\Big)\mathrm{d}s$$
$$+b(t)\int_t^{t+r(t)} \Big(a(s)x(s + h(s)) + b(s)x(s + r(s))\Big)\mathrm{d}s,$$

or equivalently,

$$\dot{x}(t) + D(t)x(t) = a(t)\int_t^{t+h(t)} E_x(s)\mathrm{d}s + b(t)\int_t^{t+r(t)} E_x(s)\mathrm{d}s. \tag{2.2}$$

Multiplying both sides of (2.2) by the factor $e^{\int_{t_0}^t D(u)\mathrm{d}u}$ and then integrating from $t_0$ to $t$, we obtain

$$x(t)e^{\int_{t_0}^t D(u)\mathrm{d}u}-x(t_0)=\int_{t_0}^t e^{\int_{t_0}^s D(u)\mathrm{d}u}a(s)\Big(\int_s^{s+h(s)}E_x(u)\mathrm{d}u\Big)\mathrm{d}s+\int_{t_0}^t e^{\int_{t_0}^s D(u)\mathrm{d}u}b(s)\Big(\int_s^{s+r(s)}E_x(u)\mathrm{d}u\Big)\mathrm{d}s,$$

which means that $x(t)$ is the solution of (2.1).

The lemma is proved. □

**Theorem 2.3** Assume that the following conditions hold,

$$\lim_{t\to\infty}\int_{t_0}^t D(s)\mathrm{d}s=\infty, \tag{2.3}$$

$$\sup_{t\geq t_0}\int_{t_0}^t e^{-\int_s^t D(u)\mathrm{d}u}\Big(|a(s)|\int_s^{s+h(s)}(|a(u)|+|b(u)|)\mathrm{d}u+|b(s)|\int_s^{s+r(s)}(|a(u)|+|b(u)|)\mathrm{d}u\Big)\mathrm{d}s:=\alpha<1. \tag{2.4}$$

Then, any solution $\{x(t),t\geq t_0\}$ of (1.1) converges to zero, that is,

$$\lim_{t\to\infty}x(t)=0.$$

**Proof** Let $\{x^*(t),t\geq t_0\}$ be an arbitrary solution (1.1). We then can define $x_0:=x^*(t_0)$. Thanks to Lemma 2.2, we know that $\{x^*(t),t\geq t_0\}$ is a solution of equation (2.1) with a initial condition $x(t_0)=x_0$. As a consequence, in order to obtain the desired result, it is enough to show that equation (2.1) with the initial condition $x(t_0)=x_0$ has an unique solution and this solution converges to zero as $t$ tends to $\infty$.

Denote by $\mathcal{C}$ the space of bounded continuous functions $x(t)$ on $[t_0,\infty)$ such that $x(t_0)=x_0$. It is seen that $\mathcal{C}$ is a complete metric space with metric

$$\rho(x,y)=\sup_{t\geq t_0}|x(t)-y(t)|.$$

We define the operator $\mathcal{P}$ on $\mathcal{C}$ as

$$(\mathcal{P}x)(t)=x_0e^{-\int_{t_0}^t D(u)\mathrm{d}u}+\int_{t_0}^t e^{-\int_s^t D(u)\mathrm{d}u}a(s)\Big(\int_s^{s+h(s)}E_x(u)\mathrm{d}u\Big)\mathrm{d}s+\int_{t_0}^t e^{-\int_s^t D(u)\mathrm{d}u}b(s)\Big(\int_s^{s+r(s)}E_x(u)\mathrm{d}u\Big)\mathrm{d}s,\quad t\geq t_0. \tag{2.5}$$

Obviously, we have $\mathcal{P}(\mathcal{C})\subset\mathcal{C}$. Let $x,y\in\mathcal{C}$, then $x(t_0)=y(t_0)=x_0$ and hence, we have

$$|(\mathcal{P}x)(t)-(\mathcal{P}y)(t)|\leq\int_{t_0}^t e^{-\int_s^t D(u)\mathrm{d}u}|a(s)|\Big(\int_s^{s+h(s)}|E_x(u)-E_y(u)|\mathrm{d}u\Big)\mathrm{d}s+\int_{t_0}^t e^{-\int_s^t D(u)\mathrm{d}u}|b(s)|\Big(\int_s^{s+r(s)}|E_x(u)-E_y(u)|\mathrm{d}u\Big)\mathrm{d}s,\quad t\geq t_0, \tag{2.6}$$

where

$$\begin{aligned}|E_x(u)-E_y(u)|&\leq|a(u)||x(u+h(u))-y(u+h(u))|+|b(u)||x(u+r(u))-y(u+r(u))|\\&\leq(|a(u)|+|b(u)|)\rho(x,y).\end{aligned}$$

As a consequence, we have

$$|(\mathcal{P}x)(t)-(\mathcal{P}y)(t)| \le \Big[\int_{t_0}^{t} e^{-\int_s^t D(u)\mathrm{d}u}\Big(|a(s)|\int_s^{s+h(s)}(|a(u)|+|b(u)|)\mathrm{d}u + |b(s)|\int_s^{s+r(s)}(|a(u)|+|b(u)|)\mathrm{d}u\Big)\mathrm{d}s\Big]\rho(x,y),\ t\ge t_0.$$

This combine with (2.4) yields

$$\rho((\mathcal{P}x),(\mathcal{P}y)) \le \alpha\rho(x,y).$$

As $\alpha<1$, we can conclude that $\mathcal{P}$ is a contractive operator.

We now consider a closed subspace $\mathcal{S}$ of $\mathcal{C}$ :

$$\mathcal{S}=\{x\in\mathcal{C} : \lim_{t\to\infty} x(t)=0\}.$$

We claim that $\mathcal{P}(\mathcal{S})\subset\mathcal{S}$. Indeed, let $x\in\mathcal{S}$, then we have

$$\begin{aligned}|(\mathcal{P}x)(t)| \ &\le\ |x_0|e^{-\int_{t_0}^t D(u)\mathrm{d}u} + \int_{t_0}^t e^{-\int_s^t D(u)\mathrm{d}u}|a(s)|\Big(\int_s^{s+h(s)}|E_x(u)|\mathrm{d}u\Big)\mathrm{d}s\\ &\quad + \int_{t_0}^t e^{-\int_s^t D(u)\mathrm{d}u}|b(s)|\Big(\int_s^{s+r(s)}|E_x(u)|\mathrm{d}u\Big)\mathrm{d}s\\ &:= I_1+I_2+I_3,\ \ t\ge t_0, \end{aligned} \tag{2.7}$$

where

$$I_1=|x_0|e^{-\int_{t_0}^t D(u)\mathrm{d}u},\quad I_2=\int_{t_0}^t e^{-\int_s^t D(u)\mathrm{d}u}|a(s)|\Big(\int_s^{s+h(s)}|E_x(u)|\mathrm{d}u\Big)\mathrm{d}s,$$

and

$$I_3=\int_{t_0}^t e^{-\int_s^t D(u)\mathrm{d}u}|b(s)|\Big(\int_s^{s+r(s)}|E_x(u)|\mathrm{d}u\Big)\mathrm{d}s.$$

By (2.3) we obtain $I_1\to 0$ as $t\to\infty$. Moreover, it follows from the fact $x\in\mathcal{S}$ that for any $\varepsilon>0$, there exists $T\ge t_0$ such that $|x(t)|<\frac{\varepsilon}{2}$ for all $t\ge T$. Hence, we have

$$\begin{aligned} I_2 &= \int_{t_0}^T e^{-\int_s^t D(u)\mathrm{d}u}|a(s)|\Big(\int_s^{s+h(s)}|E_x(u)|\mathrm{d}u\Big)\mathrm{d}s\\ &\quad + \int_{T}^t e^{-\int_s^t D(u)\mathrm{d}u}|a(s)|\Big(\int_s^{s+h(s)}|E_x(u)|\mathrm{d}u\Big)\mathrm{d}s\\ &< \int_{t_0}^T e^{-\int_s^t D(u)\mathrm{d}u}|a(s)|\Big(\int_s^{s+h(s)}|E_x(u)|\mathrm{d}u\Big)\mathrm{d}s\\ &\quad +\frac{\varepsilon}{2}\int_{T}^t e^{-\int_s^t D(u)\mathrm{d}u}|a(s)|\Big(\int_s^{s+h(s)}(|a(u)|+|b(u)|)\mathrm{d}u\Big)\mathrm{d}s,\ \ t\ge T. \end{aligned} \tag{2.8}$$

We observe that the first term in the right hand side of (2.8) converges to zero as $t\to\infty$ due to condition (2.3). Thus, there exists $T_1\ge T$, such that

$$I_2<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\int_{T}^t e^{-\int_s^t D(u)\mathrm{d}u}|a(s)|\Big(\int_s^{s+h(s)}(|a(u)|+|b(u)|)\mathrm{d}u\Big)\mathrm{d}s,\ t\ge T_1.$$

Using (2.4) we get $I_2<\varepsilon$ for all $t\ge T_1$. In other words, we have $I_2\to 0$ as $t\to\infty$.

Similarly, we also have $I_3\to 0$ as $t\to\infty$. Hence, $(\mathcal{P}x)(t)\to 0$ as $t\to\infty$.

In summary, $\mathcal{P}$ is a contractive operator and $\mathcal{P}(\mathcal{S}) \subset \mathcal{S}$. By the contraction mapping principle, $\mathcal{P}$ has a unique fixed point $x(t)$ in $\mathcal{S}$, that is,

$$x(t) = (\mathcal{P}x)(t) \text{ and } \lim_{t\to\infty} x(t) = 0.$$

This means that equation (2.1) has a unique solution and this solution satisfies $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Let us now recall a fundamental concept (see, for instance, [3]) that will be used in the next theorem.

**Definition 2.4** The ordinary differential equation $\dot{x}(t) + D(t)x(t) = 0, t \geq t_0$ is called exponentially stable, if there exist $M_0 > 0$ and $\lambda_0 > 0$ such that any solution of the equation,

$$\dot{x}(t) + D(t)x(t) = 0, t \geq s, \;\; x(s) = x_0$$

has the estimate

$$|x(t)| \leq M_0|x_0|e^{-\lambda_0(t-s)}, \;\; \forall\, t \geq s \geq t_0,$$

where $M_0$ and $\lambda_0$ do not depend on $s$.

**Theorem 2.5** Assume that $a(t)$ and $b(t)$ are bounded on $[t_0, \infty)$, that the equation $\dot{x}(t) + [a(t) + b(t)]x(t) = 0$ is exponentially stable, and that (2.4) holds. Then, any solution $x(t)$ of (1.1) exponentially converges to zero, that is, there exist constants $M, \lambda > 0$ such that

$$|x(t)| \leq Me^{-\lambda t}, \;\; \forall\, t \geq t_0.$$

**Proof** We consider the space $\mathcal{C}$ and the operator $\mathcal{P}$ as in Theorem 2.3. Let us define another closed subspace of $\mathcal{C}$ as

$$\mathcal{M} = \{x \in \mathcal{C} : \text{ there exist constants } M, \lambda > 0 \text{ such that } |x(t)| \leq Me^{-\lambda t} \;\forall\; t \geq t_0\}.$$

We will show that $\mathcal{P}(\mathcal{M}) \subset \mathcal{M}$. Let $x \in \mathcal{M}$ and use the terms $I_1, I_2, I_3$ as in (2.7).

Because equation $\dot{x}(t)+[a(t)+b(t)]x(t) = 0$ is exponentially stable, it follows from Definition 2.4 that there exist constants $M_0, \lambda_0 > 0$ such that

$$e^{-\int_s^t D(u)\mathrm{d}u} \leq M_0 e^{-\lambda_0(t-s)}, \; \forall\; t \geq s \geq t_0,$$

where $D(u) = a(u) + b(u)$. Without loss of generality, we may assume that $\lambda_0 \neq \lambda$ for $\lambda$ as in the definition of $\mathcal{M}$. We therefore have, for $I_1$ in (2.7),

$$I_1 \leq M_0|x_0|e^{\lambda_0 t_0}e^{-\lambda_0 t}, \; t \geq t_0.$$

To estimate $I_2$ in (2.7), we observe that $h(t), r(t) \geq 0$, and hence,

$$\begin{aligned}
\int_s^{s+h(s)} |E_x(u)|\mathrm{d}u &\leq M\int_s^{s+h(s)} \big(|a(u)|e^{-\lambda(u+h(u))} + |b(u)|e^{-\lambda(u+r(u))}\big)\mathrm{d}u \\
&\leq M(\overline{a} + \overline{b})\int_s^{s+h(s)} e^{-\lambda u}\mathrm{d}u \\
&= \frac{M(\overline{a} + \overline{b})}{\lambda}e^{-\lambda s}(1 - e^{-\lambda h(s)}) \\
&\leq \frac{M(\overline{a} + \overline{b})}{\lambda}e^{-\lambda s}, \; s \geq t_0,
\end{aligned}$$

where $\overline{a}=\sup_{t\geq t_0}|a(t)|$ and $\overline{b}=\sup_{t\geq t_0}|b(t)|$. Consequently, we have

$$\begin{aligned}I_2 &\leq \frac{MM_0\overline{a}(\overline{a}+\overline{b})}{\lambda}\int_{t_0}^t e^{-\lambda_0(t-s)}e^{-\lambda s}\mathrm{d}s\\ &\leq \frac{(1\vee e^{-(\lambda-\lambda_0)t_0})MM_0\overline{a}(\overline{a}+\overline{b})}{\lambda(\lambda\vee\lambda_0-\lambda\wedge\lambda_0)}e^{-(\lambda\wedge\lambda_0)t},\ t\geq t_0.\end{aligned}$$

The second inequality in the above estimates holds because if $\lambda<\lambda_0$, then,

$$\begin{aligned}\int_{t_0}^t e^{-\lambda_0(t-s)}e^{-\lambda s}\mathrm{d}s = e^{-\lambda t}\int_{t_0}^t e^{-(\lambda_0-\lambda)(t-s)}\mathrm{d}s &\leq \frac{e^{-\lambda t}}{\lambda_0-\lambda}\\ &= \frac{e^{-(\lambda\wedge\lambda_0)t}}{\lambda\vee\lambda_0-\lambda\wedge\lambda_0},\end{aligned}$$

and if $\lambda>\lambda_0$, then,

$$\begin{aligned}\int_{t_0}^t e^{-\lambda_0(t-s)}e^{-\lambda s}\mathrm{d}s = e^{-\lambda_0 t}\int_{t_0}^t e^{-(\lambda-\lambda_0)s}\mathrm{d}s &\leq \frac{e^{-(\lambda-\lambda_0)t_0}e^{-\lambda_0 t}}{\lambda-\lambda_0}\\ &= \frac{e^{-(\lambda-\lambda_0)t_0}e^{-(\lambda\wedge\lambda_0)t}}{\lambda\vee\lambda_0-\lambda\wedge\lambda_0}.\end{aligned}$$

In the same way, for $I_3$ in (2.7), we also obtain

$$I_3 \leq \frac{(1\vee e^{-(\lambda-\lambda_0)t_0})MM_0\overline{b}(\overline{a}+\overline{b})}{\lambda(\lambda\vee\lambda_0-\lambda\wedge\lambda_0)}e^{-(\lambda\wedge\lambda_0)t},\ t\geq t_0.$$

As $|(\mathcal{P}x)(t)|\leq I_1+I_2+I_3$, we infer that there exist $M^*,\lambda^*>0$ such that

$$|(\mathcal{P}x)(t)|\leq M^*e^{-\lambda^* t}\ \forall\ t\geq t_0,$$

which points out that $\mathcal{P}(\mathcal{M})\subset\mathcal{M}$.

The remainder of the proof is similar to that of Theorem 2.3. So, we omit it here. □

**Remark 2.6** In the context of stability problems for delay differential equations, it is often required that the coefficients are non-negative (see [3]). It is interesting to emphasize that Theorems 2.3 and 2.5 hold without requiring such conditions, that is, $a(t)$ and $b(t)$ may have variable-signs.

The first two theorems provide sufficient conditions for (exponential) convergence of the solution to zero. Let us now give a necessary condition for convergence of the solution to zero.

**Theorem 2.7** Assume that (2.4) and the following condition hold

$$\liminf_{t\to\infty}\int_{t_0}^t D(s)\mathrm{d}s>-\infty. \tag{2.9}$$

If all the solutions of (1.1) converge to zero, then (2.3) holds.

**Proof** Suppose that (2.3) fails. As (2.9) holds, this implies that

$$K:=\sup_{t\geq t_0}e^{-\int_{t_0}^t D(u)\mathrm{d}u}<\infty \tag{2.10}$$

and that there exists a sequence $\{t_n\}$ with $t_n\to\infty$ as $n\to\infty$ such that the sequence $\{\int_{t_0}^{t_n}D(s)\mathrm{d}s\}_{n\geq 1}$ converges to a finite constant. So, we can choose a positive constant $H$ satisfying

$$-H\leq\int_{t_0}^{t_n}D(s)\mathrm{d}s\leq H,\ \forall\ n\geq 1. \tag{2.11}$$

For the convenience of the statement, we put

$$g(s) := |a(s)| \int_s^{s+h(s)} (|a(u)| + |b(u)|)\mathrm{d}u + |b(s)| \int_s^{s+r(s)} (|a(u)| + |b(u)|)\mathrm{d}u.$$

Then, it follows from (2.4) that

$$\int_{t_0}^{t_n} e^{\int_{t_0}^{s} D(u)\mathrm{d}u} g(s)\mathrm{d}s \le \alpha e^{\int_{t_0}^{t_n} D(u)\mathrm{d}u} < e^H, \ \forall\ n \ge 1.$$

The sequence $A_n := \{\int_{t_0}^{t_n} e^{\int_{t_0}^{s} D(u)\mathrm{d}u} g(s)\mathrm{d}s\}_{n\ge 1}$ is bounded, so it has a convergent subsequence. For brevity in notation, we can assume that $\lim\limits_{n\to\infty} A_n = l$ for some $l$. Consequently, for any $\varepsilon_0 > 0$, there exists $n_0 \ge 1$ such that

$$\int_{t_{n_0}}^{t_n} e^{\int_{t_0}^{s} D(u)\mathrm{d}u} g(s)\mathrm{d}s < \frac{\varepsilon_0}{2K}, \tag{2.12}$$

for $K$ as in (2.10).

We replace $x_0$ by $\varepsilon_0$ and $t_0$ by $t_{n_0}$ in equation (2.1) to get the following equation

$$\begin{aligned} x(t) = \varepsilon_0 e^{-\int_{t_{n_0}}^{t} D(u)\mathrm{d}u} &+ \int_{t_{n_0}}^{t} e^{-\int_s^t D(u)\mathrm{d}u} a(s)\bigg(\int_s^{s+h(s)} E_x(u)\mathrm{d}u\bigg)\mathrm{d}s \\ &+ \int_{t_{n_0}}^{t} e^{-\int_s^t D(u)\mathrm{d}u} b(s)\bigg(\int_s^{s+r(s)} E_x(u)\mathrm{d}u\bigg)\mathrm{d}s, \ \ t \ge t_{n_0}. \end{aligned} \tag{2.13}$$

From Lemma 2.2, we known that the unique solution $\overline{x}(t)$ of (2.13) is also a solution of (1.1) on $[t_{n_0}, \infty)$. Using the relation

$$x(t_{n_0}) - x(t) + \int_t^{t_{n_0}} a(s)x(s+h(s))\mathrm{d}s + \int_t^{t_{n_0}} b(s)x(s+r(s))\mathrm{d}s = 0,$$

we can construct a solution $x(t)$ of (1.1) on $[t_0, \infty)$ with $x(t) = \overline{x}(t), t \ge t_{n_0}$. Because all solutions of (1.1) converge to zero, we have

$$\lim_{t\to\infty} \overline{x}(t) = 0. \tag{2.14}$$

We now fixe an $\varepsilon_0 > 0$ such that $\varepsilon_0 < \frac{1-\alpha}{Ke^H}$. From equation (2.13), we can obtain

$$\sup_{t\ge t_{n_0}} |\overline{x}(t)| \le K|\varepsilon_0| e^{\int_{t_0}^{t_{n_0}} D(u)\mathrm{d}u} + \alpha \sup_{t\ge t_{n_0}} |\overline{x}(t)| \le K|\varepsilon_0|e^H + \alpha \sup_{t\ge t_{n_0}} |\overline{x}(t)|,$$

which yields

$$\sup_{t\ge t_{n_0}} |\overline{x}(t)| \le \frac{Ke^H\varepsilon_0}{1-\alpha} < 1.$$

Hence, an application of the inequality $a + b + c \ge |a| - |b| - |c|$ to (2.13) gives us

$$\overline{x}(t_n) \ge \varepsilon_0 e^{-\int_{t_{n_0}}^{t_n} D(u)\mathrm{d}u} - \int_{t_{n_0}}^{t_n} e^{-\int_s^{t_n} D(u)\mathrm{d}u} g(s)\mathrm{d}s, \ n \ge n_0.$$

This combine with (2.12) and (2.11) implies that

$$\begin{aligned} \overline{x}(t_n) &\ge e^{-\int_{t_{n_0}}^{t_n} D(u)\mathrm{d}u}\bigg(\varepsilon_0 - e^{-\int_{t_0}^{t_{n_0}} D(u)\mathrm{d}u} \int_{t_{n_0}}^{t_n} e^{\int_{t_0}^{s} D(u)\mathrm{d}u} g(s)\mathrm{d}s\bigg) \\ &\ge e^{-\int_{t_{n_0}}^{t_n} D(u)\mathrm{d}u}\bigg(\varepsilon_0 - K\int_{t_{n_0}}^{t_n} e^{\int_{t_0}^{s} D(u)\mathrm{d}u} g(s)\mathrm{d}s\bigg) \end{aligned}$$

$$\geq e^{-\int_{t_{n_0}}^{t_n} D(u)\mathrm{d}u}\Big(\varepsilon_0 - K\frac{\varepsilon_0}{2K}\Big) \geq \frac{\varepsilon_0 e^{-2H}}{2},$$

which contradicts with (2.14).

The proof is completed. $\square$

## 3 Generalization and an Example

We observe that the method used in Section 2 does not depend on the number of advanced arguments. So, our results can be extended to the following general advanced differential equations with several terms

$$\dot{x}(t) + \sum_{k=1}^{N} a_k(t)x(t + h_k(t)), t \geq t_0. \tag{3.1}$$

where $a_k(t)$ and $h_k(t)$ are continuous functions and $h_k(t) \geq 0$.

Indeed, as in Lemma 2.2, we can rewrite (3.1) as follows:

$$x(t) = x_0 e^{-\int_{t_0}^{t} \overline{D}(u)\mathrm{d}u} + \int_{t_0}^{t} e^{-\int_s^t \overline{D}(u)\mathrm{d}u} \sum_{k=1}^{N} \Big(a_k(s) \int_s^{s+h_k(s)} \overline{E}_x(u)\mathrm{d}u\Big)\mathrm{d}s, \ \ t \geq t_0,$$

where $\overline{D}(t) = \sum\limits_{k=1}^{N} a_k(t)$ and $\overline{E}_x(t) = \sum\limits_{k=1}^{N} a_k(t)x(t + h_k(t))$. Then, similar to Section 2, we can get the following theorem without new difficulties.

**Theorem 3.1** Suppose that the following condition holds,

$$\sup_{t\geq t_0} \int_{t_0}^{t} e^{-\int_s^t \overline{D}(u)\mathrm{d}u} \sum_{k=1}^{N} \Big(|a_k(s)| \int_s^{s+h_k(s)} \sum_{i=1}^{N} |a_i(u)|\mathrm{d}u\Big)\mathrm{d}s := \alpha < 1.$$

**I** If

$$\lim_{t\to\infty} \int_{t_0}^{t} \overline{D}(s)\mathrm{d}s = \infty,$$

then any solution $\{x(t), t \geq t_0\}$ of (3.1) converges to zero.

**II** If $a_k(t), k = 1, \cdots, N$ are bounded and the equation $\dot{x}(t) + \sum\limits_{k=1}^{N} a_k(t)x(t) = 0$ is exponentially stable, then any solution $\{x(t), t \geq t_0\}$ of (3.1) exponentially converges to zero.

**Example** Consider the equation

$$\dot{x}(t) + a\sin(t)x(t + \pi) + b(1 + \cos t)x(t + \sin^2 t) = 0, \ \ t \geq 0, \tag{3.2}$$

where $a, b > 0$.

It is clear that the equation $\dot{x}(t) + [a\sin(t) + b(1 + \cos t)]x(t) = 0$ is exponentially stable. Moreover, we have the following estimates,

$$\int_s^{s+\pi} (|a\sin u| + b(1 + \cos u)) \leq a\pi + 2 + b(\pi + 2),$$

$$\int_s^{s+\sin^2 s} (|a\sin u| + b(1 + \cos u)) \leq a + 2b.$$

Applying Theorem 2.5, we can affirm that the solution of (3.2) exponentially converges to zero if

$$\sup_{t\geq 0} \int_0^t e^{-\int_s^t [a\sin u + b(1+\cos u)]\mathrm{d}u} \Big(a[a\pi + 2 + b(\pi + 2)]|\sin s| + b[a + 2b](1 + \cos s)\Big)\mathrm{d}s < 1. \tag{3.3}$$

For example, the above inequality holds if

$$\frac{e^{2a+2b}}{b}\bigg(a[a\pi+2+b(\pi+2)]+2b[a+2b]\bigg)<1,$$

which is satisfied with $a=0.01, b=0.1$.

## References

[1] Aronson D G, Weinberger H F. Nonlinear diffusion in population genetics, combustion and nerve propagation//Lecture Notes in Mathematics. Vol 446. Berlin: Springer-Verlag, 1975: 5–49

[2] Bellman R, Cooke K L. Differential-difference equations. New York-London: Academic Press, 1963

[3] Berezansky L, Braverman E. On exponential stability of a linear delay differential equation with an oscillating coefficient. Appl Math Lett, 2009, **22**(12): 1833–1837

[4] Berezansky L, Braverman E. On nonoscillation of advanced differential equations with several terms. Abstr Appl Anal, 2011, Art. ID 637142, 14 pp

[5] Burton T A, Furumochi T. Fixed points and problems in stability theory for ordinary and functional differential equations. Dynam Systems Appl, 2001, **10**(1): 89–116

[6] Britton N F. Spatial structures and periodic traveling waves in an integro-differential reaction diffusion population model. SIAM J Appl Math, 1990, **50**: 1663–1688

[7] Cohen M A, Grossberg S. Absolute stability of global pattern formation and parallel memory storage by competitive neural networks. IEEE Trans Systems Man Cybernet, 1983, **13**: 815–826

[8] Dubois D M. Extension of the Kaldor-Kalecki model of business cycle with a computational anticipated capital stock. Journal of Organisational Transformation and Social Change, 2004, **1**(1): 63–80

[9] Gabor R V. Successive approximations for the solution of second order advanced differential equations. Carpathian J Math, 2006, **22**(1/2): 57–64

[10] Jankowski T. Advanced differential equations with nonlinear boundary conditions. J Math Anal Appl, 2005, **304**(2): 490–503

[11] Kaddar A, Talibi Alaoui H. Fluctuations in a mixed IS-LM business cycle model. Electronic Journal of Differential Equations, 2008, **2008**(134): 1–9

[12] Kitamura Y, Kusano T. Oscillation of first-order nonlinear differential equations with deviating arguments. Proc Amer Math Soc, 1980, **78**(1): 64–68

[13] Kordonis I -G, Philos Ch G. Oscillation and nonoscillation in delay or advanced differential equations and in integrodifferential equations. Georgian Math J, 1999, **6**(3): 263–284

[14] Ladas G, Stavroulakis I P. Oscillations caused by several retarded and advanced arguments. Journal of Differential Equations, 1982, **44**(1): 134–152

[15] Levin S A. Dispersion and population interaction. Amer Natur, 1974, **108**: 207–228

[16] Li X, Zhu D. Oscillation and nonoscillation of advanced differential equations with variable coefficients. J Math Anal Appl, 2002, **269**(2): 462–488

[17] Myschkis A D. Lineare Differentialgleichungen mit nacheilendem Argument. Berlin: (German) Deutscher Verlag der Wissenschaften, 1955

[18] Pontryagin L S, Gamkreledze R V, Mischenko E F. The mathematical theory of optimal processes. New York-London: Interscience Publishers John Wiley & Sons, Inc, 1962

[19] Shah S M, Wiener J. Advanced differential equations with piecewise constant argument deviations. Internat J Math Math Sci, 1983, **6**(4): 671–703

[20] Schulman L S. Correlating arrows of time. Phys Rev, 1973, **7D**: 2868–2874

[21] Wiener J, Debnath L, Shah S M. Analytic solutions of nonlinear neutral and advanced differential equations. Internat J Math Math Sci, 1986, **9**(2): 365–372

[22] Pravica D W, Randriampiry N, Spurr M J. Applications of an advanced differential equation in the study of wavelets. Appl Comput Harmon Anal, 2009, **27**(1): 2–11

[23] Pravica D W, Randriampiry N, Spurr M J. Theta function identities in the study of wavelets satisfying advanced differential equations. Appl Comput Harmon Anal, 2010, **29**(2): 134–155