TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 3, pp. 835-848, June 2014
DOI: 10.11650/tjm.18.2014.3288
This paper is available online at http://journal.taiwanmathsoc.org.tw

JACOBI PROCESSES DRIVEN BY FRACTIONAL BROWNIAN MOTION

Nguyen Tien Dung

Abstract. In this paper we study a Jacobi equation driven by fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$. We first prove the existence and uniqueness of the solution. Then we investigate Malliavin differentiability and smoothness of the density of the solution. Finally, we point out that the solution can be approximated by semimartingales.

1. INTRODUCTION

It is known that the classical Jacobi process is defined as the solution of the scalar stochastic differential equation

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t(1 - X_t)}dW_t, \tag{1.1}$$

where a, b, σ are positive constants with $a < b$ and W_t is a standard Brownian motion, and plays an important role in various applications. In population biology the Jacobi process is well known as Wright-Fisher diffusion with migration studied by Karlin and Taylor (1981) [9]. In the finance context, the Jacobi process have been used by Delbaen and Shirakawa (2002) [1] to model interest rates, by De Jong et al. (2001) [8], and by Larsen and Sørensen (2007) [10] to model the exchange rates in a target zone. The Jacobi processes have also been studied by Gouriéroux and Jasiak (2006) [5], they introduced a multidimensional version and pointed out several applications.

The Jacobi process is a Markov process. However, in the last decades, many observations show that an asset price or an interest rate is not always a Markov process since it has long-range aftereffects. And in fact, many studies have pointed out that the dynamics driven by fractional Brownian motion (fBm) are a suitable choice to model such objects. We refer the reader to [3] for a short survey on applications of fBm in finance.

Received May 13, 2013, accepted November 13, 2013.
Communicated by Yuh-Jia Lee.
2010 *Mathematics Subject Classification*: 60H10, 60G22, 60H07.
Key words and phrases: Jacobi processes, Fractional Brownian motion, Malliavin calculus.
This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2011.12 (16-101).

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t, t \geq 0\}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets. On this probability space, a fBm with Hurst index $H \in (0,1)$ is a centered Gaussian process $W^H = \{W^H(t), t \geq 0\}$ with covariance function:

$$R(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

Moreover, fBm with $H > \frac{1}{2}$ has the following Volterra representation

$$W_t^H = \int_0^t K(t,s)dW_s, \tag{1.2}$$

where W_t is a standard Brownian motion and the kernel $K(t,s), t \geq s,$ is given by

$$K(t,s) = c_H \int_s^t \frac{u^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(u-s)^{H-\frac{3}{2}}du,$$

where c_H is a standardized constant depending only on H.

Naturally, it would be desirable to study the fractional Jacobi processes, that is, to replace Brownian motion in the equation (1.1) by a fBm:

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t(1-X_t)}dW_t^H,\ t \in [0,T], \tag{1.3}$$

where W_t^H is fBm with $H > \frac{1}{2}$. Recently, there are many papers that are devoted to the problems of the existence and uniqueness of the solution of stochastic differential equations driven by fBm (see, for instance, [4, 7, 12, 13] and the references therein). Unfortunately, since the volatility coefficient of (1.3), $\sigma\sqrt{x(1-x)}$, is only $\frac{1}{2}$-Hölder continuous, we cannot apply these known results to (1.3). On the other hand, unlike the equation (1.1) we cannot also apply the Yamada-Watanabe condition [14] to (1.3) because fBm is neither a Markov process nor a semimartingale, except for $H = \frac{1}{2}$.

Because of the complexity of the fractional stochastic calculus, the literature concerning fractional stochastic differential equations with non-Lipschitz, especially $\frac{1}{2}$-Hölder continuous, volatility coefficient is rare. And the aim of this paper is to study the fractional Jacobi equation (1.3). More specifically, we obtain the following results

(1) The existence and uniqueness of the solution.

(2) The solution belongs to $(0,1)$ for any initial value $X_0 \in [0,1]$. Moreover, if $X_0 \in (0,1)$ then there exists $\varepsilon > 0$ such that $X_t \in [\varepsilon, 1-\varepsilon]$ for any $t \in [0,T]$.

(3) Malliavin differentiability of the solution and smoothness of the density with respect to Lebesgue measure on $\mathbb{R}$.

(4) The solution can be approximated by semimartingales.

In order to make this equation having a sense, let us define its solution. A strong solution of (1.3) is a stochastic process X with sample paths in the space $C[0,T]$ of continuous functions from $[0,T]$ to the interval $[0,1]$ and has a form for all $t \in [0,T]$

$$X_t = X_0 + \int_0^t (a - bX_s)ds + \sigma \int_0^t \sqrt{X_s(1-X_s)}dW_s^H,$$

the initial condition $X_0 \in [0,1]$ is a constant, the integral $\int_0^t \sqrt{X_s(1-X_s)}dW_s^H$ should be interpreted as a pathwise Riemann-Stieltjes integral. We refer the reader to the paper of Zähle [15] for a detailed presentation of this integral. Here, we will just recall some basic concepts.

Fix a parameter $0 < \lambda < \frac{1}{2}$, denote by $W^{1-\lambda,\infty}[0,T]$ the space of measurable function $g : [0,T] \to \mathbb{R}$ such that

$$\|g\|_{1-\lambda,\infty} := \sup_{0\le s<t\le T} \Big(\frac{|g(t)-g(s)|}{(t-s)^{1-\lambda}} + \int_s^t \frac{|g(y)-g(s)|}{(y-s)^{2-\lambda}}dy \Big) < +\infty,$$

and by $W^{\lambda,1}[0,T]$ the space of measurable function $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|_{\lambda,1} := \int_0^T \frac{|f(s)|}{s^\lambda}ds + \int_0^T \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\lambda+1}}dsdt < \infty\,.$$

For the functions $f \in W^{\lambda,1}[0,T], g \in W^{1-\lambda,\infty}[0,T]$, Zähle introduced the generalized Stieltjes integral

$$\int_0^T f(t)dg(t) = (-1)^\lambda \int_0^T D_{0+}^\lambda f(t) D_{T-}^{1-\lambda} g(t)dt$$

defined in terms of the fractional derivative operators

$$D_{0+}^\lambda f(t) = \frac{1}{\Gamma(1-\lambda)} \left(\frac{f(t)}{t^\lambda} + \lambda \int_0^t \frac{f(t)-f(y)}{(t-y)^{\lambda+1}}dy \right),$$

$$D_{T-}^\lambda g(t) = \frac{(-1)^\lambda}{\Gamma(1-\lambda)} \left(\frac{g(t)-g(T)}{(T-t)^\lambda} + \lambda \int_t^T \frac{g(t)-g(y)}{(y-t)^{\lambda+1}}dy \right).$$

Denote by $C^\lambda[0,T]$ the space of Hölder continuous functions of order λ. We have the following change-of-variable formula.

Proposition 1.1. *If $h \in C^\mu[0,T]$ and $F \in C^1(\mathbb{R})$ is a real-valued function such that $F'(h) \in C^\lambda[0,T]$ for some $\mu + \lambda > 1$, then, for any $t \in [0,T]$*

$$F(h(t)) = F(h(0)) + \int_0^t F'(h(s))dh(s).$$

Since W^H has Hölder continuous sample paths of exponent lesser than H, it belongs to $W^{1-\lambda,\infty}[0,T]$ if $H > \frac{1}{2}$ and $\lambda > 1-H$. Hence, the fractional stochastic integral $\int_0^t f(s)dW_s^H, t \in [0,T]$ is well defined for any function $f \in W^{\lambda,1}[0,T]$ for some λ such that $1-H < \lambda < \frac{1}{2}$.

2. The Main Results

To get desired results, we will not directly do the proof for the equation (1.3), but for a coordinate transformation thereof. We consider the process $V_t = \arcsin(2X_t - 1)$ which by the change-of-variable formula satisfies

$$dV_t = \frac{2a-b-b\sin V_t}{\cos V_t}dt + \sigma dW_t^H. \tag{2.1}$$

Thus this transformation allows us to shift the nonlinearity from the volatility coefficient into the drift coefficient. Then the results can be more easily proved.

The equation (2.1) belongs to the class of singular stochastic differential equations. For this class, we would like to mention a work made by Hu et al. [6]. They have studied the following equation

$$dV_t = V_0 + \int_0^t f(s,V_s)ds + W_t^H,\ t \geq 0,$$

where the drift $f(t,x)$ is nonnegative, it has a singularity at $x = 0$ and satisfies some suitable conditions.

Since the drift of the equation (2.1) is not always nonnegative and has two singularity points at $x = \pm\frac{\pi}{2}$, we cannot apply Hu et al.'s results to our case. However, their ideas can be effectively used in the proposition below.

Proposition 2.1. *Assume that* $0 < a < b$. *Then the equation (2.1) admits a unique solution on* $[0,\infty)$ *for any initial value* $V_0 \in [-\frac{\pi}{2},\frac{\pi}{2}]$. *Moreover,* $V_t \in (-\frac{\pi}{2},\frac{\pi}{2})$ $a.s.$ *for any* $t > 0$.

Proof. If $V_0 \in (-\frac{\pi}{2},\frac{\pi}{2})$, the function $g(x) := \frac{2a-b-b\sin x}{\cos x}$ is Lipschitz continuous on a neighborhood of V_0. Hence, there exists a local solution V_t on the interval $[0,\tau)$, where τ is the stopping time such that $\tau = \inf\{t > 0 : |V_t| = \frac{\pi}{2}\}$. Assume that $\tau < \infty$.

Case 1. $V_\tau = \frac{\pi}{2}$. For all $t \in [0,\tau)$ we have $\frac{\pi}{2} - V_t > 0$ and

$$\frac{\pi}{2} = V_\tau = V_t + \int_t^\tau g(V_s)ds + \sigma(W_\tau^H - W_t^H),$$

or equivalently

$$(2.2)\qquad \frac{\pi}{2} - V_t - \int_t^\tau g(V_s)ds + \sigma(W_t^H - W_\tau^H) = 0.$$

We observe that $\lim\limits_{x\to\frac{\pi}{2}^-} \frac{-2a+b+b\sin x}{\cos x}(\frac{\pi}{2} - x) = 2b - 2a$. Hence, there exists $\varepsilon > 0$ such that

$$\frac{-2a+b+b\sin x}{\cos x} > \frac{b-a}{\frac{\pi}{2} - x}\ \forall\ x \in (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}).$$

Since V_t is continuous and $V_\tau = \frac{\pi}{2}$, there exists t_0 such that $V_t \in (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2})\ \forall\ t \in [t_0, \tau)$ which implies that

$$(2.3)\qquad -g(V_t) > \frac{b-a}{\frac{\pi}{2} - V_t} > 0\ \forall\ t \in [t_0, \tau).$$

Recall that the paths of fBm are β-Hölder continuous for any $\beta < H$. Thus, if we fix $\beta \in (\frac{1}{2}, H)$ then there exists a finite random variable $C_\beta(\omega)$ such that $\sigma|W_\tau^H - W_t^H| \leq C_\beta(\omega)(\tau - t)^\beta$. Combining (2.2) and (2.3) gives us

$$\frac{\pi}{2} - V_t < C_\beta(\omega)(\tau - t)^\beta\ \forall\ t \in [t_0, \tau),$$

and

$$-\int_t^\tau g(V_s)ds < C_\beta(\omega)(\tau - t)^\beta\ \forall\ t \in [t_0, \tau).$$

As a consequence, it follows from (2.3) that

$$C_\beta(\omega)(\tau - t)^\beta > -\int_t^\tau g(V_s)ds > \int_t^\tau \frac{b-a}{\frac{\pi}{2} - V_s}ds > \frac{b-a}{(1-\beta)C_\beta(\omega)}(\tau - t)^{1-\beta}\ \forall\ t \in [t_0, \tau).$$

Therefore

$$(1-\beta)C_\beta(\omega)^2 > (b-a)(\tau - t)^{1-2\beta}\ \forall\ t \in [t_0, \tau)$$

which is a contradiction because the right hand side of the above inequality tends to ∞ as $t \to \tau$. We conclude that $\tau = \infty$.

Case 2. $V_\tau = -\frac{\pi}{2}$. For all $t \in [0, \tau)$ we also have $V_t + \frac{\pi}{2} > 0$ and

$$V_t + \frac{\pi}{2} + \int_t^\tau g(V_s)ds + \sigma(W_\tau^H - W_t^H) = 0.$$

Similarly, there exists t_0 such that

$$g(V_t) > \frac{a}{\frac{\pi}{2} + V_t} > 0\ \forall\ t \in [t_0, \tau).$$

Once again, we get a contradiction and conclude that $\tau = \infty$.

Thus we already proved the existence of global solution to (2.1) with $V_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. If $V_0 = \frac{\pi}{2}$, for each $n \geq 1$ we denote by V_t^n the solution to (2.1) with the initial condition $V_0^n = \frac{\pi}{2} - \frac{1}{n}$. Obviously, $\{V_t^n\}_{n\geq 1}$ is a increasing sequence and $|V_t^n| < \frac{\pi}{2}$. Hence, it has a limit, denoted by V_t. By the monotone convergence theorem V_t satisfies

$$V_t = \frac{\pi}{2} + \int_0^t g(V_s)ds + \sigma W_t^H,\ t \geq 0$$

Hence, $g(V_t) < \infty$ for almost all $t \geq 0$, and this implies that $|V_t| < \frac{\pi}{2}$ for almost all $t \geq 0$. By the previous arguments, if $|V_t| < \frac{\pi}{2}$, then $|V_s| < \frac{\pi}{2}$ for all $s > t$. As a consequence, $|V_t| < \frac{\pi}{2}$ for all $t \geq 0$.

The case of $V_0 = -\frac{\pi}{2}$ can be proved similarly.

In order to prove the pathwise uniqueness we assume that $V_t^\star$ and V_t^* are two solutions with $V_0^\star = V_0^*$. We have

$$d(V_t^\star - V_t^*) = [g(V_t^\star) - g(V_t^*)]dt$$

and hence

$$(V_t^\star - V_t^*)^2 = 2\int_0^t (V_s^\star - V_s^*)[g(V_s^\star) - g(V_s^*)]ds.$$

Noting that $g'(x) = \frac{(2a-b)\sin x - b}{\cos^2 x} < 0\ \forall\ x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. These, together with Lagrange's theorem, imply

$$(V_t^\star - V_t^*)^2 \leq 0\ \forall\ t \geq 0.$$

The Proposition is proved. ■

Theorem 2.1. *Assume that $0 < a < b$ and the initial condition $X_0 \in [0, 1]$. Then the Jacobi equation (1.3) has a unique solution in $C^{H^-}[0, T] = \bigcap_{\beta<H} C^\beta[0, T]$. Moreover, this solution belongs to $(0, 1)$ and is Malliavin differentiable with*

$$D_s^W X_t = \sigma\sqrt{X_t(1 - X_t)} \int_s^t \partial_1 K(v, s) e^{\int_v^t \frac{(2a-b)X_u - a}{2X_u(1-X_u)}du} dv\,,\ t \geq s, \tag{2.4}$$

where $\partial_1 K(t, s) = \frac{\partial}{\partial t} K(t, s)$.

Before giving a proof of the above theorem, let us recall some elements of Malliavin calculus with respect to Brownian motion W (we refer the reader to [11] for more details about this topic). The complete probability space $(\Omega, \mathcal{F}, P)$ is now associated to a Brownian motion W which is used to present fBm W^H as in (1.2).

For $h \in L^2([0, T], \mathbb{R})$, we denote by $W(h)$ the Wiener integral

$$W(h) = \int_0^T h(t)dW_t.$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega, \mathcal{F}, P)$ consisting of those classes of random variables of the form

$$F = f(W(h_1), ..., W(h_n)), \tag{2.5}$$

where $n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n, L^2([0,T],\mathbb{R})), h_1, ..., h_n \in L^2([0,T],\mathbb{R})$. If F has the form (2.5), we define its derivative as the process $D^W F := \{D_t^W F, t \in [0,T]\}$ given by

$$D_t^W F = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(W(h_1), ..., W(h_n))h_k(t).$$

More generally, for each $k \geq 1$ we can define the iterated derivative operator on a cylindrical random variable by setting

$$D_{t_1,...,t_k}^{W,k} F = D_{t_1}^W ... D_{t_k}^W F.$$

For any $1 \leq p < \infty$, we shall denote by $\mathbb{D}_W^{1,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} := \big[E|F|^p\big]^{\frac{1}{p}} + E\Bigg[\int_0^T |D_u^W F|^p du\Bigg]^{\frac{1}{p}}.$$

Proof of the Theorem 2.1. We first prove the existence of the solution. It is easy to see that the solution V_t of (2.1) is β-Hölder continuous on $[0,T]$ for some $\beta \in (\frac{1}{2}, H)$. Consequently, by applying Proposition 1.1 to $h(t) = V_t$ and $F(h) = \frac{\sin h+1}{2}$ we obtain that $X_t := \frac{\sin V_t+1}{2} \in C^{H^-}[0,T]$ is a solution to (1.3) for any initial condition $X_0 \in [0,1]$. Obviously, $X_t \in (0,1)$ because $V_t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. In order to show that this solution is unique in $C^{H^-}[0,T]$, let us consider Y_t is another solution in $C^{H^-}[0,T]$ with the same initial condition $Y_0 = X_0$. The following arguments are the same as in an example given in [15, Section 5]. If $Y_0 \in (0,1)$ then there exist constants $0 < c < C < 1$ such that $c < Y_t < C$ for $0 < t \leq \varepsilon$ with suÂciently small $\varepsilon > 0$. For these t we can apply Proposition 1.1 to $h(t) = Y_t, t \in [0,\varepsilon]$ and to $F(h) = \arcsin(2h-1), h \in (c,C)$ and obtain $U_t := \arcsin(2Y_t - 1)$ is a solution of the equation (2.1) on $[0,\varepsilon]$. Since the solution of (2.1) is unique, we can infer that $Y_t = X_t$ for $t \in [0,\varepsilon]$. In the same way we can show that for any $t > 0$ with $Y_t = X_t$ there exists a right-sided neighborhood where the functions coincide. So we can conclude that $Y_t = X_t$ on $[0,T]$. If $Y_0 \in \{0,1\}$, we also get the same conclusion by using a generalized change-of-variable formula which was presented in [16, Theorem 3.1].

We now are in a position to prove the solution X_t is Malliavin differentiable. To do this, it enough to show that V_t is Malliavin differentiable.

We first consider the case, where $|V_0| < \frac{\pi}{2}$. Because the function $g(x)$ is not differentiable at $x = \pm\frac{\pi}{2}$, we need to approximate it by a continuously differentiable function on $\mathbb{R}$. For each $n \geq 1$, let $\chi_n(x)$ be a continuously differentiable function satisfying $\chi_n(x) \leq 1 \;\forall\; x$ and

$$\chi_n(x) = \begin{cases} 0, & |x| > \frac{\pi}{2} - \frac{1}{n} \\ 1, & |x| \leq \frac{\pi}{2} - \frac{2}{n}. \end{cases}$$

Put $g_n(x) := g(x)\chi_n(x)$ with $g_n(x) = 0$ at x such that $\cos x = 0$. We have $g_n(x)$ is bounded and continuously differentiable satisfying

$$g_n'(x) = \begin{cases} 0, & |x| > \frac{\pi}{2} - \frac{1}{n} \\ g'(x), & |x| \le \frac{\pi}{2} - \frac{2}{n}. \end{cases}$$

Consider the "approximation" equation

$$dV_t^{(n)} = g_n(V_t^{(n)})dt + \sigma dW_t^H \tag{2.6}$$

with the initial condition $V_0^{(n)} = V_0$.

Define a stopping time $\tau_n = \inf\{t > 0 : |V_t| > \frac{\pi}{2} - \frac{1}{n}\}$. Clearly, $\{\tau_n\}_{n\ge 1}$ is an increasing sequence. Since $|V_t| < \frac{\pi}{2},\ \forall\, t \ge 0$, this implies that $\lim\limits_{n\to\infty} \tau_n = \infty$.

By the definition of the function $\chi_n(x)$ we have

$$V_t^{(n)} = V_{\tau_{\frac{n}{2}} \wedge t}\,,\ \forall\, t \ge 0.$$

Therefore, for each $t \ge 0$ we have

$$\lim_{n\to\infty} V_t^{(n)} = \lim_{n\to\infty} V_{\tau_{\frac{n}{2}} \wedge t} = V_t \ a.s.$$

By the bounded convergence theorem we have $V_t^{(n)}$ converges in $L^2(\Omega)$ to V_t as n tends to ∞.

Since $g_n(x)$ is a continuously differentiable function on $\mathbb{R}$, we can apply the chain rule of Malliavin derivative (see, [11, Theorem 2.2.1]) to (2.6) and obtain

$$\frac{dD_s^W V_t^{(n)}}{dt} = g_n'(V_t^{(n)}) D_s^W V_t^{(n)} + \sigma \partial_1 K(t,s)\,,\ t \ge s,$$

subject to the boundary condition $D_s^W V_s^{(n)} = K(s,s) = 0$. Solving the above equation gives us

$$D_s^W V_t^{(n)} = \sigma \int_s^t \partial_1 K(v,s) e^{\int_v^t g_n'(V_u^{(n)})du} dv\,,\ t \ge s.$$

Obviously, $D_s^W V_t^{(n)}$ converges to $\sigma \int_s^t \partial_1 K(v,s) e^{\int_v^t g'(V_u)du} dv$. Moreover, $D_s^W V_t^{(n)} \le \sigma K(t,s)$. Once again, from the bounded convergence theorem we conclude that $D_s^W V_t^{(n)}$ converges in $L^2(\Omega)$ to $\sigma \int_s^t \partial_1 K(v,s) e^{\int_v^t g'(V_u)du} dv$. Then by closability of Malliavin derivative we have

$$D_s^W V_t = \sigma \int_s^t \partial_1 K(v,s) e^{\int_v^t g'(V_u)du} dv\,,\ t \ge s.$$

Thus V_t is Malliavin differentiable for any the initial value $V_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

We now consider $V_0 = \frac{\pi}{2}$. As in Proposition 2.1, we denote by $V_t^n, n \geq 1$ the solution to (2.1) with the initial condition $V_0^n = \frac{\pi}{2} - \frac{1}{n}$. Since $V_0^n < \frac{\pi}{2}$, this implies that V_t^n is Malliavin differentiable and furthermore, we have

$$D_s^W V_t^n = \sigma \int_s^t \partial_1 K(v,s) e^{\int_v^t g'(V_u^n)du} dv\,,\ t \geq s.$$

By the almost sure convergence of V_t^n to the solution V_t of (2.1) combined with the fact $|V_t^n| < \frac{\pi}{2}\ \forall\ n \geq 1$, we obtain $V_t^n \to V_t$ in $L^2(\Omega)$ as $n \to \infty$. Noting that $g'(x) < 0\ \forall\ x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then by using the bounded convergence theorem and then the closability of Malliavin derivative we can conclude for the case of $V_0 = \frac{\pi}{2}$ that

$$D_s^W V_t = \sigma \int_s^t \partial_1 K(v,s) e^{\int_v^t g'(V_u)du} dv.$$

The case of $V_0 = -\frac{\pi}{2}$ is proved similarly.

We now turn our attention to Malliavin differentiability of X_t. But this is obvious because $X_t = \frac{\sin V_t + 1}{2}$. The expression (2.4) follows from the relation $D_s^W X_t = \frac{1}{2}\cos V_t D_s^W V_t$.

The Theorem is proved.

Corollary 2.1. *Under the assumptions of Theorem 2.1. If $\sigma > 0$, then for any $t \in (0,T]$ the law of X_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. Follows directly from from the expression (2.4) and Theorem 2.1.3 in [11]. ∎

Next, we discuss the smoothness of the density of the solution. We will need the following technical lemma.

Lemma 2.1. *Suppose that $|V_0| < \frac{\pi}{2}$, then there exists $\varepsilon > 0$ such that the solution V_t of the equation (2.1) satisfies $|V_t| \leq \frac{\pi}{2} - \varepsilon$ for all $t \geq 0$.*

Proof. It is obvious that there exists $\varepsilon_1 > 0$ such that $|V_0| < \frac{\pi}{2} - \varepsilon_1$. Moreover, by the definition of the function $g(x)$ there exists $\varepsilon_2 > 0$ such that

$$-g(x) > 0\ \forall\ x \in (\frac{\pi}{2} - \varepsilon_2, \frac{\pi}{2}) \text{ and } g(x) > 0\ \forall\ x \in (-\frac{\pi}{2}, -\frac{\pi}{2} + \varepsilon_2).$$

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, we will show that $|V_t| \leq \frac{\pi}{2} - \varepsilon\ a.s.$ for all $t \geq 0$.

Denote $\tau_1 = \inf\{t > 0 : |V_t| = \frac{\pi}{2} - \varepsilon\}$. If $\tau_1 = \infty$, then the proof is complete. When $\tau_1 < \infty$ we have $|V_{\tau_1}| = \frac{\pi}{2} - \varepsilon$. Without loss of generality, we can assume that $V_{\tau_1} = \frac{\pi}{2} - \varepsilon$.

We first show that $V_t \leq \frac{\pi}{2} - \varepsilon\ a.s.$ for all $t \geq \tau_1$. Indeed, suppose that there exists $t_1 > \tau_1$ such that $V_{t_1} > \frac{\pi}{2} - \varepsilon\ a.s.$ Because V_t is continuous and $V_{t_1} > V_{\tau_1}$, there exist $t_2 < t_3$ such that V_t is increasing on $(t_2, t_3) \subset (\tau_1, t_1)$. Thus $\frac{\pi}{2} - \varepsilon < V_{t_2} \leq V_t \leq V_{t_3} < \frac{\pi}{2}$ for all $t \in (t_2, t_3)$. This implies that $-g(V_s) > 0$ for all $s \in (t_2, t_3)$. Moreover, we have

$$V_{t_3} - V_{t_2} - \int_{t_2}^{t_3} g(V_s)ds = \sigma(W^H_{t_3} - W^H_{t_2}),$$

which gives us a contradiction because

$$0 = \sigma E(W^H_{t_3} - W^H_{t_2}) = E\bigg(V_{t_3} - V_{t_2} - \int_{t_2}^{t_3} g(V_s)ds\bigg) > 0.$$

The remaining of the proof is to show that $V_t \geq -\frac{\pi}{2} + \varepsilon \ a.s.$ for all $t \geq \tau_1$. Denote $\tau_2 = \inf\{t > 0 : V_t = -\frac{\pi}{2} + \varepsilon\} > \tau_1$. If $\tau_2 < \infty$, following the same lines as the above arguments we obtain that $V_t \geq -\frac{\pi}{2} + \varepsilon \ a.s.$ for all $t \geq \tau_2$.

The Lemma is proved. ■

Theorem 2.2. *Assume that* $0 < a < b$ *and* $\sigma > 0$. *If the initial condition* $X_0 \in (0,1)$, *then, for any* $t \in (0,T]$ *the solution* X_t *of the Jacobi equation (1.3) has an infinitely differentiable density with respect to Lebesgue measure on* $\mathbb{R}$.

Proof. Fix $t \in (0,T]$, to apply the Malliavin criterion for the existence of a smooth density, we have to check:

(i) $X_t \in \mathbb{D}^\infty = \bigcap\limits_{i\geq 1} \bigcap\limits_{p\geq 1} \mathbb{D}^{i,p}_W$,

(ii) $\big(\int_0^t |D^W_s X_t|^2 ds\big)^{-1} \in \bigcap\limits_{p\geq 1} L^p(\Omega)$.

Since $X_0 \in (0,1)$, it follows from Lemma 2.1 that there exists $\varepsilon > 0$ such that $X_t \in [\varepsilon, 1-\varepsilon] \subset (0,1)$ for all $t \in [0,T]$. As a consequence, it is easy to check by using the formula (2.4) that $X_t \in \mathbb{D}^\infty$.

In order to prove the condition (ii) it is enough to check that for any $p \geq 1$ there exists $\theta_0 > 0$ such that

$$P\bigg(\int_0^t |D^W_s X_t|^2 ds \leq \theta\bigg) \leq \theta^p, \text{ for all } 0 < \theta < \theta_0.$$

For convenience of statement, let us put $m(x) = \frac{(2a-b)x-a}{2x(1-x)}, x \in (0,1)$. By using the integration by parts formula we have

$$D^W_s X_t = \sigma\sqrt{X_t(1-X_t)}\bigg(K(t,s) + \int_s^t K(v,s)m(X_v)e^{\int_v^t m(X_u)du} dv\bigg).$$

Since $X_t \in [\varepsilon, 1-\varepsilon]$, we can infer that

$$\sigma^2 X_t(1-X_t) \geq \sigma^2 \varepsilon(1-\varepsilon) := n_\varepsilon > 0.$$

Hence,

$$|D_s^W X_t|^2 \geq n_\varepsilon \left(K(t,s) + \int_s^t K(v,s) m(X_v) e^{\int_v^t m(X_u)du} dv \right)^2.$$

Let $\alpha, \beta > 0$, we can write

$$P\left(\int_0^t |D_s^W X_t|^2 ds \leq \theta \right) \leq P\left(\int_{t-2\theta^\alpha}^{t-\theta^\alpha} |D_s^W X_t|^2 ds \leq \theta \right) \leq P_{1,\theta} + P_{2,\theta},$$

with

$$\begin{aligned} P_{1,\theta} &= P\Bigg(n_\varepsilon \int_{t-2\theta^\alpha}^{t-\theta^\alpha} \left(K(t,s) + \int_s^t K(v,s) m(X_v) e^{\int_v^t m(X_u)du} dv \right)^2 ds \leq \theta, \\ &\qquad \sup_{t-2\theta^\alpha < s < t-\theta^\alpha} \int_s^t K(v,s) |m(X_v)| e^{\int_v^t m(X_u)du} dv \leq \theta^\beta \Bigg), \\ P_{2,\theta} &= P\left(\sup_{t-2\theta^\alpha < s < t-\theta^\alpha} \int_s^t K(v,s) |m(X_v)| e^{\int_v^t m(X_u)du} dv > \theta^\beta \right). \end{aligned}$$

To estimate $P_{1,\theta}$ we noting that $K(t,s) \geq \frac{c_H}{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}}$, $t \geq s$. Then when $\beta > (H - \frac{1}{2})\alpha$ we obtain

$$\begin{aligned} P_{1,\theta} &\leq P\left(n_\varepsilon \int_{t-2\theta^\alpha}^{t-\theta^\alpha} \left(\frac{c_H}{H-\frac{1}{2}} \theta^{(H-\frac{1}{2})\alpha} - \theta^\beta \right)^2 ds \leq \theta \right) \\ &= P\left(n_\varepsilon \left(\frac{c_H}{H-\frac{1}{2}} \theta^{(H-\frac{1}{2})\alpha} - \theta^\beta \right)^2 \theta^\alpha \leq \theta \right). \end{aligned}$$

Choosing α such that $\alpha < \frac{1}{2H}$, it is clear that $P_{1,\theta} = 0$.

We now use Chebyshev's inequality to get for any $q > 1$

$$\begin{aligned} P_{2,\theta} &\leq \frac{1}{\theta^{q\beta}} E\left(\sup_{t-\theta^\alpha < s < t-\theta^\alpha} \left| \int_s^t K(v,s) m(X_v) e^{\int_v^t m(X_u)du} dv \right|^q \right) \\ &\leq \frac{m_\varepsilon^q}{\theta^{q\beta}} \sup_{t-2\theta^\alpha < s < t-\theta^\alpha} \left| \int_s^t K(v,s) dv \right|^q, \end{aligned}$$

where $m_\varepsilon = \sup\limits_{\varepsilon \le x \le 1-\varepsilon} |m(x)| < \infty$. On the set $\{(s,v) : t - 2\theta^\alpha < s < t - \theta^\alpha, s < v < t\}$ we have

$$K(v,s) = c_H \int\limits_s^v \frac{u^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(u-s)^{H-\frac{3}{2}}du \le \frac{c_H}{H-\frac{1}{2}} \frac{t^{H-\frac{1}{2}}}{(t-2\theta^\alpha)^{H-\frac{1}{2}}}(v-s)^{H-\frac{1}{2}}.$$

Therefore we have

$$\begin{aligned} P_{2,\theta} &\le \frac{m_\varepsilon^q}{\theta^{q\beta}}\left[\frac{c_H}{(H-\frac{1}{2})(H+\frac{1}{2})}\frac{t^{H-\frac{1}{2}}}{(t-2\theta^\alpha)^{H-\frac{1}{2}}}\right]^q (2\theta)^{q(H+\frac{1}{2})\alpha} \\ &= \left[\frac{c_H m_\varepsilon}{(H-\frac{1}{2})(H+\frac{1}{2})}\frac{t^{H-\frac{1}{2}}}{(t-2\theta^\alpha)^{H-\frac{1}{2}}} 2^{(H+\frac{1}{2})\alpha}\right]^q \theta^{q(H+\frac{1}{2})\alpha - q\beta} \\ &\le \left[\frac{2^{H-\frac{1}{2}} c_H m_\varepsilon}{(H-\frac{1}{2})(H+\frac{1}{2})} 2^{\frac{1}{2}+\frac{1}{4H}}\right]^q \theta^{q(H+\frac{1}{2})\alpha - q\beta} \end{aligned}$$

for all $\theta \le \theta_0$, where $\theta_0 > 0$ such that $\theta_0^\alpha < \frac{t}{4}$.

So, choosing β such that $(H-\frac{1}{2})\alpha < \beta < (H+\frac{1}{2})\alpha < \frac{H+\frac{1}{2}}{2H}$, the proof is complete. ■

We end this paper with a interesting result which says that our non-semimartingale dynamical system (1.3) can be approximated by semimartingales. Before doing this, let us recall the semimartingale approximation of fBm: For every $\delta > 0$ we define

$$W_t^{H,\delta} := \int\limits_0^t K(t+\delta, s)dW_s,\ t \in [0,T].$$

It is well known from [2] that $W_t^{H,\delta}$ is a semimartingale with the following decomposition

$$W_t^{H,\delta} = \int\limits_0^t \varphi_s^\delta ds + \int\limits_0^t K(s+\delta,s)dW_s, \tag{2.7}$$

where $\varphi_s^\delta = \int\limits_0^s \partial_1 K(s+\delta,u)dW_u$. Moreover, $W_t^{H,\delta}$ converges in $L^p(\Omega),\ p \ge 1$ uniformly in $t \in [0,T]$ to W_t^H as $\delta \to 0:$

$$E|W_t^{H,\delta} - W_t^H|^p \le c_p \delta^{pH}. \tag{2.8}$$

Theorem 2.3. *Assume that* $0 < a < b$. *Then the solution* X_t *of the Jacobi equation (1.3) with the initial condition* $X_0 \in (0,1)$ *can be approximated in* $L^p(\Omega), p \ge 1$ *by semimartingales.*

Proof. We first consider the equation

$$dV_t^\delta = \frac{2a-b-b\sin V_t^\delta}{\cos V_t^\delta}dt + \sigma dW_t^{H,\delta},\ V_0^\delta = V_0. \tag{2.9}$$

We have $|V_0| < \frac{\pi}{2}$ because $0 < X_0 < 1$. Consequently, a similar proof to one of Lemma 2.1 results in $|V_t^\delta| \leq \frac{\pi}{2} - \varepsilon$ for all t.

On the interval $[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon]$, the function $g(x) = \frac{2a-b-b\sin x}{\cos x}$ is Lipchitz and its derivative is bounded by a positive constant M_ε. Let V_t be the solution of (2.1), then from (2.8) and by using Gronwall's lemma we can check that

$$E|V_t^\delta - V_t|^p \leq 2^{p-1}c_p\delta^{pH}e^{2^{p-1}M_\varepsilon^p t},\ \forall\ t \in [0,T].$$

As a consequence, we have the following convergence uniformly in $t \in [0,T]$

$$X_t^\delta := \frac{\sin V_t^\delta + 1}{2} \to X_t = \frac{\sin V_t + 1}{2} \text{ in } L^p(\Omega) \text{ as } \delta \to 0.$$

From the decomposition (2.7) we see that the equation (2.9) is an Itô stochastic differential equation. By using Itô formula we get

$$\begin{aligned} dX_t^\delta &= \left(a + \frac{1}{4}\sigma^2K^2(t+\delta,t) - (b+\frac{1}{2}\sigma^2K^2(t+\delta,t))X_t^\delta + \sigma\varphi_t^\delta\sqrt{X_t^\delta(1-X_t^\delta)}\right)dt \\ &\quad + \sigma K^2(t+\delta,t)\sqrt{X_t^\delta(1-X_t^\delta)}dW_t, \end{aligned}$$

which implies that X_t^δ is a semimartingale.

The Theorem is proved. ■

Acknowledgments

The author would like to thank the anonymous referees for their valuable comments which led to improvement of this work.

References

1. F. Delbaen and H. Shirakawa, An interest rate model with upper and lower bounds, *Asia Pac Fin Mark*, **9(3)** (2002), 191-209.
2. N. T. Dung, Semimartingale approximation of fractional Brownian motion and its applications, *Comput. Math. Appl.*, **61(7)** (2011), 1844-1854.
3. N. T. Dung, Mackey-Glass equation driven by fractional Brownian motion, *Physica A*, **391** (2012), 5465-5472.
4. N. T. Dung, Fractional stochastic differential equations with applications to finance, *J. Math. Anal. Appl.*, **397(1)** (2013), 334-348.
5. C. Gouriéroux and J. Jasiak, Multivariate Jacobi process with application to smooth transitions, *J. Econ.*, **131(1-2)** (2006), 475-505.

6. Y. Hu, D. Nualart and X. Song, A singular stochastic differential equation driven by fractional Brownian motion, *Statist. Probab. Lett.*, **78(14)** (2008), 2075-2085.

7. Yu-J. Jien and J. Ma, Stochastic differential equations driven by fractional Brownian motions, *Bernoulli*, **15(3)** (2009), 846-870.

8. F. De Jong, F. C. Drost and B. J. M. Werker, A jump-diffusion model for exchange rates in a target zone, *Statistica Neerlandica*, **55** (2001), 270-300.

9. S. Karlin and H. M. Taylor, *A Second Course in Stochastic Processes*, Academic Press, New York, 1981.

10. K. Larsen and M. Sørensen, Diffusion models for exchange rates in a target zone, *Math. Fin.*, **17(2)** (2007), 285-306.

11. D. Nualart, *The Malliavin Calculus and Related Topics*, 2nd edition, Springer, 2006.

12. D. Nualart and A. Răşcanu, Differential equations driven by fractional Brownian motion, *Collect. Math.*, **53** (2002), 55-81.

13. D. Nualart and B. Saussereau, Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion, *Stoch. Proc. Appl.*, **119** (2009), 391-409.

14. T. Yamada and S. Watanabe, On the uniqueness of solutions of stochastic differential equations, *J. Math. Kyoto Univ*, **11** (1971), 155-167.

15. M. Zähle, Integration with respect to fractal functions and stochastic calculus, Part I, *Probab. Theory Related Fields*, **111** (1998), 333-374.

16. M. Zähle, Integration with respect to fractal functions and stochastic calculus Part II, *Math. Nachr.*, **225** (2001), 145-183.

Nguyen Tien Dung
Department of Mathematics
Fpt University
No. 8 Ton That Thuyet
My Dinh, Tu Liem, Hanoi
VietNam
E-mail: dung_nguyentien10@yahoo.com
dungnt@fpt.edu.vn