ON THE LIMIT CLOSURE OF A SEQUENCE OF ELEMENTS IN LOCAL RINGS

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Abstract. We present a systematic study about the limit closure $(\underline{x})^{\lim}$ of a sequence of elements $\underline{x}$ (eg. a system of of parameters) in a local ring. Firstly, we answer the question which elements are always contained in the limit closure of a system of parameters. Then we apply this result to give a characterization of systems of parameters which is a generalization of previous results of Dutta and Roberts in [12] and of Fouli and Huneke in [13]. We also prove a topological characterization of unmixed local rings. In two dimensional case, we compute explicitly the limit closure of a system of parameters. Some interesting examples are given.

Contents

1. Introduction

Parameter ideals and systems of parameters are basic concepts of local algebra. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension t and $\underline{x} = x_1, \ldots, x_t$ a system of parameters. Understanding the relations of elements in a system of parameters is one of the most important problems in commutative algebra. Indeed, Hochster asked about a "simple" relation that cannot be satisfied by a system of parameters (cf. [17]). This question is called the monomial conjecture and stated as follows. For for all $n \geq 1$ we have $(x_1 \ldots x_t)^n \notin (x_1^{n+1}, \ldots, x_t^{n+1})$.

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The monomial conjecture has affirmative answer when R contains a field or $\dim R \leq 3$, but it is wild open in other cases (cf. [18, 16]). Moreover it is easy to see that

$$(x_1^2,\dots,x_t^2):(x_1\dots x_t) \subseteq (x_1^3,\dots,x_t^3):(x_1\dots x_t)^2 \subseteq \cdots \subseteq (x_1^{n+1},\dots,x_t^{n+1}):(x_1\dots x_t)^n \subseteq \cdots.$$

Thus it is natural to consider the following

$$(x_1,\dots,x_t)^{\lim} := \bigcup_{n>0}\big((x_1^{n+1},\dots,x_t^{n+1}):(x_1...x_t)^n\big).$$

The monomial conjecture is equivalent to the claim that 1 *cannot* be contained in $(x_1,\dots,x_t)^{\lim}$ for any system of parameters $\underline{x} = x_1,\dots,x_t$. We call $(x_1,\dots,x_t)^{\lim}$ (or $(\underline{x})^{\lim}$) the *limit closure* of the sequence $\underline{x} = x_1,\dots,x_t$.

It is worth to note that if R is Cohen-Macaulay then $(x_1,\dots,x_t)^{\lim} = (x_1,\dots,x_t)$ and the converse holds true (cf. [15, 8]). The motivation of our paper is a question which can be thought of as the opposite of Hochster's monomial conjecture: Determine elements which are *always* contained in $(x_1,\dots,x_t)^{\lim}$ for all systems of parameters $\underline{x} = x_1,\dots,x_t$? For convenience we shall consider this problem for modules. Let $(R,\mathfrak{m})$ be a Noetherian local ring, M a finitely generated R-modules of dimension d. Let $\underline{x} = x_1,\dots,x_r$ be a sequence of r elements of R. Then the *limit closure* of the sequence $\underline{x}$ in M is a submodule of M defined by

$$(\underline{x})_M^{\lim} = \bigcup_{n>0}\big((x_1^{n+1},\dots,x_r^{n+1})M:(x_1...x_r)^n\big).$$

The following problem is the starting point of this work.

Problem 1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring, M a finitely generated R-modules of dimension d. What is*

$$\bigcap_{\underline{x}}(x_1,\dots,x_d)_M^{\lim},$$

where $\underline{x} = x_1,\dots,x_d$ runs over all systems of parameters of M?

We will show that the above intersection can be interpreted by the primary decomposition of the zero submodule of M. Let $(0) = \cap_{\mathfrak{p}\in \operatorname{Ass} M} N(\mathfrak{p})$ be a reduced primary decomposition of the zero submodule of M. The *unmixed component* $U_M(0)$ of M is a submodule defined by

$$U_M(0) = \bigcap_{\mathfrak{p}\in \operatorname{Ass} M, \dim R/\mathfrak{p}=d} N(\mathfrak{p})$$

It should be noted that $U_M(0)$ is just the largest submodule of M of dimension less than $\dim M = d$. We settle Problem 1 as follows.

Theorem 1.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension d. Let $\underline{x} = x_1,...,x_d$ be a system of parameters of M. Then*

$$\bigcap_{n>0}(x_1^n,\dots,x_d^n)_M^{\lim} = U_M(0).$$

Along the way we also consider the intersection of limit closures of parts of systems of parameters. We prove the following, for the definitions of *dimension filtration* and *good* system of parameters see Section 4.

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension d with the dimension filtration $\mathcal{D} : H^0_{\mathfrak{m}}(M) = D_0 \subseteq D_1 \cdots \subseteq D_{t-1} \subseteq D_t = M$, where $\dim D_i = d_i$ for all $i = 0, \ldots t$. Let $\underline{x} = x_1, \ldots, x_d$ be a good system of parameters of M. Then we have*

$$\bigcap_{n \geq 1} (x_1^n, \ldots, x_j^n)_M^{\lim} = D_i.$$

for all $0 \leq i \leq t-1$ and $d_i < j \leq d_{i+1}$.

These intersection formulas and their variations play an important role in this paper. We found many applications of them for some deep problems. Firstly, we generalize the previous works of Dutta and Roberts [12] and of Fouli and Huneke [13] about relation which can *only* be satisfied by systems of parameters. Let $\underline{x} = x_1, \ldots, x_t$ be a system of parameter of R and $\underline{y} = y_1, \ldots, y_t$ a sequence of elements such that $(\underline{y}) \subseteq (\underline{x})$. We have a matrix $A = (a_{ij})$, $a_{ij} \in R$, $1 \leq i, j \leq r$ such that $y_i = \sum_{j=1}^n a_{ij} x_j$, it means $\mathbf{y} = A\mathbf{x}$, where $\mathbf{x}$ (res. $\mathbf{y}$) denotes the column vector with entries $x_1, \ldots, x_r$ (res. $y_1, \ldots, y_r$). We abbreviate it by writing $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$. It easily follows from Crammer's rule that $\det(A).(\underline{x}) \subseteq (\underline{y})$. Therefore, we obtain a determinantal map

$$\det(A) \,:\, R/(\underline{x}) \to R/(\underline{y}), \quad m + (\underline{x}) \mapsto \det(A)m + (\underline{y}).$$

When R is Cohen-Macaulay, Dutta and Roberts in [12] proved that $\underline{y}$ is a system of parameters if and only if the determinantal map $\det(A)$ is injective. In [13] Fouli and Huneke extended this result for any local ring where they substituted $(\underline{x})$ by $(\underline{x})^{\lim}$. By [30, 5.1.15] we also have a homomorphism

$$\det(A) \,:\, R/(\underline{x})^{\lim} \to R/(\underline{y})^{\lim},$$

which is independent of the choice of the matrix A. The following is a generalization (and a correction) of Fouli and Huneke's result.

Theorem 1.3. *Let $(R, \mathfrak{m})$ be a catenary equidimensional local ring of dimension t. There exists a positive integer ℓ, which depends only on R, with property: whenever $\underline{x} = x_1, ..., x_t$ is a system of parameters of R with $(\underline{x}) \subseteq \mathfrak{m}^\ell$ and $\underline{y} = y_1, \ldots, y_t$ a sequence of elements such that $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$ the following statements are equivalent*

(i) *$\underline{y}$ forms a system of parameters of R.*

(ii) *The determinantal map $R/(\underline{x})^{\lim} \xrightarrow{\det A} R/(\underline{y})^{\lim}$ is injective.*

As another application of intersection formulas we give a characterization of *unmixed* local rings. In local algebra we often pass to the $\mathfrak{m}$-adic completion $\widehat{R}$ to inherit many good properties of complete local rings. A local ring $(R, \mathfrak{m})$ is called *unmixed* (in the sense of Nagata) if $\dim \widehat{R}/\mathfrak{P} = \dim \widehat{R}$ for all $\mathfrak{P} \in \operatorname{Ass} \widehat{R}$. Almost local domains in commutative algebra are unmixed. However Nagata in [25, Example 2, pp. 203–205] constructed an local domain with $U_{\widehat{R}}(0) \neq 0$ that is R is not unmixed. The following is a surprising characterization of unmixed local rings in terms of the topology defined by limit closures.

Theorem 1.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension t. Then R is unmixed if and only if the $\mathfrak{m}$-adic topology is equivalent to the topology defined by $\{(x_1^n, \ldots, x_t^n)^{\lim}\}_{n \geq 1}$ for any system of parameters $x = x_1, \ldots, x_t$ of R.*

It should be noted that the limit closure is very complicate to compute. In fact by Heitmann's work on monomial conjecture we (only) know $(\underline{x})^{\lim} \subseteq \mathfrak{m}$ or $\ell(R/(\underline{x})^{\lim}) > 0$ for any system of parameters $\underline{x}$ when $\dim R$ is at most three. In the two last sections of this paper we give some explicit computations for limit closures. By the intersection formulas we always can reduce to the case R is unmixed. When $\dim R = 2$ we prove the following result.

Theorem 1.5. *Let $(R, \mathfrak{m})$ be an unmixed local ring of dimension $d = 2$ with the S_2-ification S. Let x, y be a system of parameters R. Then we have the following.*

(i) $(x, y)^{\lim} = (x, y)S \cap R$.
(ii) $\ell(R/(x, y)^{\lim}) = e(x, y; R) - \ell(H^1_{\mathfrak{m}}(R)/(x, y)H^1_{\mathfrak{m}}(R))$, where $e(x, y; R)$ is the multiplicity of (x, y).
(iii) $\ell((x, y)^{\lim}/(x, y)) = \ell(H^1(x, y; H^1_{\mathfrak{m}}(R)))$.

We also compute the limit closure of a sequence of elements based on an example of Huneke about the Lichtenbaum-Hartshorne vanishing theorem (cf. Proposition 8.6). The paper is organized as follows.

In Section 2 we prove some basic and important properties of limit closure. The main technique is understanding the limit closure via the canonical map from Koszul cohomology to local cohomology. Then the vanishing theorems of local cohomology is very useful to prove Theorem 1.1. For convenience, we will deal the limit closure of any sequence on a finite generated R-module.

The Sections 3 and 4 are devoted for the intersection formulas of Theorems 1.1 and 1.2 and their variations.

We prove Theorem 1.3 in Section 5. We also provide an example to claim that the catenary condition of the Theorem is essential.

Section 6 is devoted for Theorem 1.4.

In Section 7 we first consider the relation between of limit closures of a system of parameters in R and in its S_2-ification. Then we apply the obtained result to prove Theorem 1.5.

In the last Section we compute an explicit example of certain limit closure.

Throughout this paper, R is a commutative Noetherian ring and M is a finitely generated R-module. The set of associated primes of M is denoted by $\operatorname{Ass} M$. We also denote $\operatorname{Assh} M = \{\mathfrak{p} \in \operatorname{Ass} M \ : \ \dim R/\mathfrak{p} = \dim M\}$. For a sequence of elements $\underline{x} = x_1, \ldots, x_r$ and a positive integer n we denote by $\underline{x}^{[n]}$ the sequence $x_1^n, \ldots, x_r^n$. About concepts of commutative algebra we follow [2, 23]. For local cohomology we refer to [1]. We also want to note that some results of this paper, including Theorems 1.1, 1.3 and 1.4, appeared in [Proceeding of the 6-th Japan-Vietnam Joint Seminar on Commutative Algebra, Hayama, Japan 2010, 127–135]. The readers are encouraged to [26] for an application of the limit closure to F-singularities.

2. Basic properties

Throughout this section, R is a Noetherian ring, and M is a finitely generated R-modules. Let $\underline{x} = x_1, \dots, x_r$ be a sequence of r elements of R. For a positive integer n, we set $\underline{x}^{[n]} = x_1^n, \dots, x_r^n$. The following is the main object of this paper.

Definition 2.1 ([20])**.** The *limit closure* of the sequence $\underline{x}$ in M is a submodule of M defined by

$$(\underline{x})_M^{\lim} = \bigcup_{n>0} \big((\underline{x}^{[n+1]})M : (x_1...x_r)^n\big),$$

when $M = R$ we write $(\underline{x})^{\lim}$ for short.

It is easy to see that

$$(\underline{x}^{[2]})M : (x_1...x_r) \subseteq (\underline{x}^{[3]})M : (x_1...x_r)^2 \subseteq \cdots \subseteq (\underline{x}^{[n+1]})M : (x_1...x_r)^n \subseteq \cdots .$$

Thus the notion of limit closure is well-defined. By the Noetherness, $(\underline{x})_M^{\lim} = (\underline{x}^{[s+1]})M : (x_1...x_r)^s$ for some $s \geq 1$.

Remark 2.2. (i) When $(R, \mathfrak{m})$ is a local ring and $x_1, \dots, x_r \in \mathfrak{m}$, it is well-known that $(\underline{x})_M^{\lim} = (\underline{x})M$ if and only if $\underline{x}$ is an M-sequence.

(ii) The notion of limit closure appears naturally when we consider local cohomology as the limit of Koszul cohomology. For a sequence $\underline{x} = x_1, \dots, x_r$. We have a direct system $\{M/(\underline{x}^{[n]})M\}_{n\geq 1}$ given by the determinantal maps

$$(x_1...x_r)^{m-n} \ : \ M/(\underline{x}^{[n]})M \longrightarrow M/(\underline{x}^{[m]})M$$

for $1 \leq n \leq m$. Then the kernel of the canonical map

$$M/(\underline{x})M \to \varinjlim M/(\underline{x}^{[n]})M \cong H^r_{(\underline{x})}(M)$$

is $(\underline{x})_M^{\lim}/(\underline{x})M$, where $H^i_{(\underline{x})}(M)$ is the i-th local cohomology of M with support in $(\underline{x})$. We get that the induced direct system $\{M/(\underline{x}^{[n]})_M^{\lim}\}_{n\geq 1}$ with injective maps and

$$\varinjlim M/(\underline{x}^{[n]})_M^{\lim} \cong H^r_{(\underline{x})}(M).$$

Therefore we can consider $M/(\underline{x}^{[n]})_M^{\lim}$ as a submodule of $H^r_{(\underline{x})}(M)$. Hence

$$\mathrm{Ann}(H^r_{(\underline{x})}(M)) = \cap_{n\geq 1}\mathrm{Ann}(M/(\underline{x}^{[n]})_M^{\lim}).$$

In particular, $\mathrm{Ann}(H^r_{(\underline{x})}(R)) = \cap_{n\geq 1}(\underline{x}^{[n]})^{\lim}$.

(iii) There is a special interest when $(R, \mathfrak{m})$ is a local ring and $\underline{x}$ is a system of parameters of R. In this case, the Hochster monomial conjecture is equivalent to say that $(\underline{x})^{\lim} \subsetneq R$ is a proper ideal of R for all systems of parameters $\underline{x}$. It is well-known that $(\underline{x})^{\lim} \subseteq \mathfrak{m}$ (or $\ell(R/(\underline{x})^{\lim}) \geq 1$) for all systems of parameters $\underline{x}$ if either R contains a field or R has dimension at most three (cf. [16, 18]). In fact, by Grothendieck's non-vanishing theorem we have $H^t_{\mathfrak{m}}(R) \neq 0$, here $t = \dim R$. According to (ii), for each system of parameters $\underline{x} = x_1, \dots, x_t$ there exists a positive integer n_0 (depends on $\underline{x}$) such that $(\underline{x}^{[n]})^{\lim} \subseteq \mathfrak{m}$ for all $n \geq n_0$.

Now, let $\underline{y} = y_1, \ldots, y_r$ be another sequence of elements such that $(\underline{y}) \subseteq (\underline{x})$. Then there exists a matrix $A = (a_{ij})$, $a_{ij} \in R$, $1 \leq i, j \leq r$ such that $y_i = \sum_{j=1}^{n} a_{ij} x_j$, it means $\mathbf{y} = A\mathbf{x}$, where $\mathbf{x}$ (res. $\mathbf{y}$) denotes the column vector with entries $x_1, \ldots, x_r$ (res. $y_1, \ldots, y_r$). Following [13], we abbreviate it by writing $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$. It easily follows from Crammer's rule that $\det(A).(\underline{x}) \subseteq (\underline{y})$. Therefore, we obtain a canonical map

$$\det(A) : M/(\underline{x})M \to M/(\underline{y})M, \quad m + (\underline{x})M \mapsto \det(A)m + (\underline{y})M.$$

By [30, 5.1.15] we also have that $\det(A).(\underline{x})_M^{\lim} \subseteq (\underline{y})_M^{\lim}$. Hence we obtain a homomorphism

$$\det(A) : M/(\underline{x})_M^{\lim} \to M/(\underline{y})_M^{\lim},$$

which is independent of the choice of the matrix A. The map $\det(A)$ is called *determinantal maps.*

Remark 2.3. Let $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$ are sequences such that $\sqrt{(\underline{x})} = \sqrt{(\underline{y})}$ i.e. $(\underline{x}^{[n]}) \overset{B}{\subseteq} (\underline{y})$ for some B and n. Then the determinantal map $\det(A) : M/(\underline{x})_M^{\lim} \to M/(\underline{y})_M^{\lim}$ is injective (cf. [8, Lemma 3.1]). Therefore $(\underline{x})_M^{\lim} = (\underline{y})_M^{\lim} :_M \det(A)$. Hence $(\underline{y})_M^{\lim} \subseteq (\underline{x})_M^{\lim}$.

The following is a slight generalization of [17, Proposition 2], and [8, Theorem 3.3].

Proposition 2.4. *Let M be a finitely generated R-modules of dimension d. Then there exists a positive integer n such that every system of parameters $\underline{y} = y_1, \ldots, y_d$ contained in $\mathfrak{m}^n$ satisfies the monomial property, i.e. $(\underline{y})_M^{\lim} \neq M$.*

Proof. Without any loss of generality we may assume that $\operatorname{Ann} M = 0$. Then by Remark 2.2 (iii) we can choose a system of parameters $\underline{x} = x_1, \ldots, x_d$ satisfies the monomial property. Therefore by Remark 2.3, the positive integer n such that $\mathfrak{m}^n \subseteq (\underline{x})$ satisfies the requirement of the proposition. □

Lemma 2.5. *Let M be a finitely generated R-modules of dimension d, and $\underline{x} = x_1, \ldots, x_r$ a sequence of elements in R. Then the following assertions hold true.*

(i) $(\underline{x})_M^{\lim} = M$ *if* $H^r_{(\underline{x})}(M) = 0$. *In particular, if $r > d$ then* $(\underline{x})_M^{\lim} = M$.
(ii) *If $r = d$ then $M/(\underline{x})_M^{\lim}$ has finite length.*

Proof. Note first that $M/(\underline{x})_M^{\lim}$ is a submodule of $H^r_{(\underline{x})}(M)$ by Remark 2.2 (ii). So the statement (i) is trivial.
(ii) If $r = d$, $H^d_{(\underline{x})}(M)$ is an Artinian module (cf. [1, Excercise 7.1.7]), and hence $M/(\underline{x})_M^{\lim}$ is both Noetherian and Artinian. Thus $\ell(M/(\underline{x})_M^{\lim}) < \infty$. □

Corollary 2.6. *Let $\underline{x} = x_1, \ldots, x_r$ be a sequence of elements, and N a submodule of M such that* $\dim N < r$. *Then $N \subseteq (\underline{x})_M^{\lim}$.*

Proof. By Lemma 2.5 we have $(\underline{x})^{\lim}_N = N$. Hence

$$\begin{aligned}(\underline{x})^{\lim}_M &= \bigcup_{n>0}\big((\underline{x}^{[n+1]})M :_M x_1^n...x_r^n\big)\\ &\supseteq \bigcup_{n>0}\big((\underline{x}^{[n+1]})N :_N x_1^n...x_r^n\big)\\ &= (\underline{x})^{\lim}_N = N.\end{aligned}$$

□

The following is very useful in the sequel.

Proposition 2.7. *Let $\underline{x} = x_1, \ldots, x_r$ be a sequence of elements, and N a submodule of M such that $N \subseteq \cap_{n>0}(\underline{x}^{[n]})^{\lim}_M$. Set $\overline{M} = M/N$. Then we have $(\underline{x})^{\lim}_{\overline{M}} = (\underline{x})^{\lim}_M/N$.*

Proof. It is sufficient to prove that

$$(\underline{x})^{\lim}_M = \bigcup_{m>0}\big(((\underline{x}^{[m+1]})M + N) :_M x_1^m...x_r^m\big).$$

In fact, the set on the left hand is clear contained in the set on the right hand. Conversely, it is easy to check that

$$((\underline{x}^{[m+1]})M + N) :_M x_1^m...x_r^m \subseteq ((\underline{x}^{[m'+1]})M + N) :_M x_1^{m'}...x_r^{m'},$$

for all $m \leq m'$. Then there exists a positive integer s such that

$$\bigcup_{m>0}\big(((\underline{x}^{[m+1]})M + N) :_M x_1^m...x_r^m\big) = ((\underline{x}^{[s+1]})M + N) :_M x_1^s...x_r^s.$$

By the assumption $N \subseteq (\underline{x}^{[s+1]})^{\lim}_M$, there exists a positive integer k such that

$$N \subseteq (x_1^{(k+1)(s+1)}, \ldots, x_r^{(k+1)(s+1)})M :_M x_1^{k(s+1)}...x_r^{k(s+1)}.$$

Therefore

$$(x_1^{s+1}, \ldots, x_r^{s+1})M + N \subseteq (x_1^{(k+1)(s+1)}, \ldots, x_r^{(k+1)(s+1)})M :_M x_1^{k(s+1)}...x_r^{k(s+1)}.$$

Thus

$$\begin{aligned}(\underline{x})^{\lim}_M &\supseteq (x_1^{(k+1)(s+1)}, \ldots, x_r^{(k+1)(s+1)})M :_M x_1^{k(s+1)+s}...x_r^{k(s+1)+s}\\ &= \big((x_1^{(k+1)(s+1)}, \ldots, x_r^{(k+1)(s+1)})M :_M x_1^{k(s+1)}...x_r^{k(s+1)}\big) :_M x_1^s...x_r^s\\ &\supseteq ((\underline{x}^{[s+1]})M + N) :_M x_1^s...x_r^s\\ &= \bigcup_{m>0}\big(((\underline{x}^{[m+1]})M + N) :_M x_1^m...x_r^m\big).\end{aligned}$$

The proof is complete. □

Proposition 2.8. *Let M be a finitely generated R-module of dimension d and $\underline{x} = x_1, \ldots, x_d$ a sequence of elements of R. Then the following conditions are equivalent*

(i) $H^d_{(\underline{x})}(M) = 0$.
(ii) $\cap_{n\geq 1}(\underline{x}^{[n]})^{\lim}_M = M$.
(iii) $\dim R/\operatorname{Ann} H^d_{(\underline{x})}(M) < d$.

Proof. (i) $\Leftrightarrow$ (ii) follows from Remark 2.2 (ii).
(i) $\Rightarrow$ (iii) is trivial.
(iii) $\Rightarrow$ (ii) Set $M' = M/\cap_{n\geq 1}(\underline{x}^{[n]})^{\lim}_M$. By Proposition 2.7 we have $\cap_{n\geq 1}(\underline{x}^{[n]})^{\lim}_{M'} = 0$. By Remark 2.2 (ii) again we have

$$\mathrm{Ann}\, H^d_{(\underline{x})}(M) = \cap_{n\geq 1}\mathrm{Ann}\big(M/(\underline{x}^{[n]})^{\lim}_M\big) = \mathrm{Ann} M'.$$

Thus $\dim R/\mathrm{Ann}\, H^d_{(\underline{x})}(M) = \dim M'$. So we have $\dim M' < t$. It follows by Lemma 2.5 that $\cap_{n\geq 1}(\underline{x'}^{[n]})^{\lim}_{M'} = M'$. Therefore $M' = 0$ and hence $\cap_{n\geq 1}(\underline{x}^{[n]})^{\lim}_M = M$. $\square$

3. Intersection of limit closures

The aim of this section is to prove Theorem 1.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension d. Recall that the *unmixed component* $U_M(0)$ of M is a submodule defined by

$$U_M(0) = \bigcap_{\mathfrak{p}\in \mathrm{Assh} M} N(\mathfrak{p}),$$

where $0 = \cap_{\mathfrak{p}\in \mathrm{Ass}\, M} N(\mathfrak{p})$ is a reduced primary decomposition of the zero submodule of M (see [11]). We need some auxiliary results from [4, 9] to prove Theorem 1.1. Let $\underline{x} = x_1, ..., x_d$ be a system of parameters of M. Then we can consider the differences

$$I_{M,\underline{x}}(n) = \ell(M/(\underline{x}^{[n]})M) - e(\underline{x}^{[n]}; M), \text{ and}$$

$$J_{M,\underline{x}}(n) = e(\underline{x}^{[n]}; M) - \ell(M/(\underline{x}^{[n]})^{\lim}_M)$$

as functions in n, where $e(\underline{x}; M)$ is the Serre multiplicity of M with respect to the sequence $\underline{x}$. In general, these functions are not polynomials in n (see [10]), but they are bounded above by polynomials. Moreover, we have

Theorem 3.1 (see, [4, 9])**.** *With the notations as above, the both functions $I_{M,\underline{x}}(n)$ and $J_{M,\underline{x}}(n)$ are non-negative increasing, and the least degrees of polynomials in n bounding above these functions are independent of the choice of $\underline{x}$. Moreover, if we denote by $p(M)$ and $pf(M)$ for these least degrees with respect to $I_{M,\underline{x}}(n)$ and $J_{M,\underline{x}}(n)$ respectively, then $p(M) \leq d-1$ and $pf(M) \leq d-2$.*

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. We set $N = \cap_{n>0}(\underline{x}^{[n]})^{\lim}_M$. By Corollary 2.6 we have $U_M(0) \subseteq N$. Put $\overline{M} = M/U_M(0)$ and $M' = M/N$. By Proposition 2.7 we have

$$\ell(M/(\underline{x}^{[n]})^{\lim}_M) = \ell(M'/(\underline{x}^{[n]})^{\lim}_{M'}) = \ell(\overline{M}/(\underline{x}^{[n]})^{\lim}_{\overline{M}})$$

for all $n \geq 1$. Then by Theorem 3.1, there are polynomials $f(n)$ of degree at most $d-1$ and $g(n)$ of degree at most $d-2$ such that

$$\ell(M/(\underline{x}^{[n]})^{\lim}_M) = \ell(M'/(\underline{x}^{[n]})^{\lim}_{M'}) \leq \ell(M'/(\underline{x}^{[n]})M') \leq n^d e(\underline{x}; M') + f(n),$$

and

$$\ell(M/(\underline{x}^{[n]})^{\lim}_M) = \ell(\overline{M}/(\underline{x}^{[n]})^{\lim}_{\overline{M}}) \geq n^d e(\underline{x}; \overline{M}) - g(n).$$

Therefore

$$f(n) + g(n) \geq n^d(e(\underline{x}, \overline{M}) - e(\underline{x}, M'))$$

for all $n > 0$. It follows that $e(\underline{x}; N/U_M(0)) = e(\underline{x}; \overline{M}) - e(\underline{x}; M') = 0$. Hence $\dim N < d$, and so $N = U_M(0)$, since $U_M(0)$ is the largest submodule of M with the dimension less than d. The proof is complete. $\square$

The following result is an immediate consequence of Theorem 1.1.

Corollary 3.2. $\bigcap_{\underline{x}} (\underline{x})_M^{\lim} = U_M(0)$, *where* $\underline{x}$ *runs through the set of all systems of parameters of* M.

Corollary 3.3. *Let* M *be a finitely generated* R*-module of dimension* d.

$$\mathrm{Ann}\,(H^d_{\mathfrak{m}}(M)) = \mathrm{Ann}\,(M/U_M(0)) = \{r \in R \,:\, \dim M/(0 :_M r) < d\}.$$

In particular, $H^d_{\mathfrak{m}}(M) \neq 0$.

Proof. Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameter of M. By Remark 2.2, we may consider $M/(\underline{x}^{[n]})_M^{\lim}$ as a submodule of $H^d_{\mathfrak{m}}(M) = H^d_{(\underline{x})}(M)$ for all $n \geq 1$.

$$\begin{aligned}
\mathrm{Ann}\,(H^d_{\mathfrak{m}}(M)) &= \{r \in R \,:\, rM \subseteq M/(\underline{x}^{[n]})_M^{\lim}, \forall\, n \geq 1\} \\
&= \{r \in R \,:\, rM \subseteq U_M(0)\} \\
&= \mathrm{Ann}\,(M/U_M(0)) \\
&= \{r \in R \,:\, \dim(rM) < d\} \\
&= \{r \in R \,:\, \dim M/(0 :_M r) < d\}.
\end{aligned}$$

The last assertion follows from the first. $\square$

The following was proved by Grothendieck [14, Proposition 6.6 (7)].

Corollary 3.4. *Let* $(R, \mathfrak{m})$ *be a complete local ring, and* M *a finitely generated* R*-module of dimension* d. *Let* $T^d(M) = \mathrm{Hom}_R(H^d_{\mathfrak{m}}(M), E(R/\mathfrak{m}))$, *where* $E(R/\mathfrak{m})$ *is the injective envelope of* $R/\mathfrak{m}$. *Then* $\mathrm{Ann}\,(T^d(M)) = U_{R/\mathrm{Ann}M}(0)$.

Proof. We may assume that $\mathrm{Ann}M = 0$. Therefore $\mathrm{Assh}M = \mathrm{Assh}R$. By duality we have $\mathrm{Ann}T^d(M) = \mathrm{Ann}H^d_{\mathfrak{m}}(M)$. So by Corollary 3.3 we need only to show that $\mathrm{Ann}\,(M/U_M(0)) = U_R(0)$ which is equivalent to

$$\{r \in R \,:\, \dim rM < d\} = \{r \in R \,:\, \dim rR < d\}.$$

Since M is a faithfully R-module then the following homomorphism

$$R \to M^k, x \mapsto (xm_1, \ldots, xm_k)$$

is injective, where $m_1, \ldots, m_k$ are generators of M. We also have a natural projective homomorphism $R^k \to M$. Thus for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and for all $r \in R$ we have $rM_{\mathfrak{p}} = 0$ iff $rR_{\mathfrak{p}} = 0$. Therefore

$$\begin{aligned}
\{r \in R \,:\, \dim rM < d\} &= \{r \in R \,:\, rM_{\mathfrak{p}} = 0, \forall\, \mathfrak{p} \in \mathrm{Assh}M\} \\
&= \{r \in R \,:\, rR_{\mathfrak{p}} = 0, \forall\, \mathfrak{p} \in \mathrm{Assh}R\} \\
&= \{r \in R \,:\, \dim rR < d\}.
\end{aligned}$$

The proof is complete. $\square$

Corollary 3.5. *Let $(R,\mathfrak{m})$ be a complete local ring, and M a finitely generated R-module of dimension d. Let $\underline{x} = x_1, \ldots, x_d$ be a sequence of elements. Let $\cap_{\mathfrak{p}\in \operatorname{Ass} M} N(\mathfrak{p}) = 0$ be a reduced primary decomposition of (0) in M. Then*

$$\bigcap_{n\geq 1} (\underline{x}^{[n]})^{\lim}_M = \bigcap_{\mathfrak{p}\in J} N(\mathfrak{p}),$$

where $J = \{\mathfrak{p} \in \operatorname{Ass} M : \underline{x}$ is a system of parameters of $R/\mathfrak{p}\}$.

Proof. Set $N = \cap_{\mathfrak{p}\in J} N(\mathfrak{p})$. Then $\operatorname{Ass} N = \operatorname{Ass} M \setminus J$. By the Lichtenbaum-Hartshorne vanishing Theorem we have $H^d_{(\underline{x})}(N) = 0$. Therefore $\cap_{n\geq 1}(\underline{x}^{[n]})^{\lim}_M \supseteq \cap_{n\geq 1}(\underline{x}^{[n]})^{\lim}_N = N$ by Proposition 2.8. Set $M' = M/N$, we have $\operatorname{Ass} M' = J \subseteq \operatorname{Assh} M$. Then $\underline{x}$ is a system of parameters of M' and $U_{M'}(0) = 0$. Hence $\cap_{n\geq 1}(\underline{x}^{[n]})^{\lim}_{M'} = 0$ by Theorem 1.1. Thus $\cap_{n\geq 1}(\underline{x}^{[n]})^{\lim}_M = N$ by Proposition 2.7. □

We prove a non-vanishing result of local cohomology.

Corollary 3.6. *Let M be a finitely generated R-module of dimension d and $x_1, \ldots, x_r$ be a part of system of parameters. Then $H^r_{(x_1,\ldots,x_r)}(M) \neq 0$.*

Proof. Extend $x_1, \ldots, x_r$ to a full system of parameters $\underline{x} = x_1, \ldots, x_d$. For all $n \geq 1$ we have $(x_1^n, \ldots, x_r^n)^{\lim}_M \subseteq (\underline{x}^{[n]})^{\lim}_M$. Therefore

$$\bigcap_{n\geq 1} (x_1^n, \ldots, x_r^n)^{\lim}_M \subseteq U_M(0)$$

by Theorem 1.1. By Remark 2.2 (ii) we have

$$\operatorname{Ann} H^r_{(x_1,\ldots,x_r)}(M) = \cap_{n\geq 1} \operatorname{Ann} M/(x_1^n, \ldots, x_r^n)^{\lim}_M \subseteq \operatorname{Ann} M/U_M(0).$$

Thus $H^r_{(x_1,\ldots,x_r)}(M) \neq 0$. The proof is complete. □

4. The dimension filtration

In this section we study the intersection of the limit closures of parts of systems of parameters. We first recall the notions of dimension filtration and good system of parameters (cf. [6, 11, 29]).

Definition 4.1. Let M be a finitely generated R-module of dimension d.

(i) The *dimension filtration* of M is the filtration of submodules

$$\mathcal{D} : H^0_{\mathfrak{m}}(M) = D_0 \subseteq D_1 \cdots \subseteq D_{t-1} \subseteq D_t = M,$$

where D_{i-1} is the largest submodule of D_i with $\dim D_{i-1} < \dim D_i$ for all $i = 1, \ldots, t$.

(ii) A system of parameters $\underline{x} = x_1, \ldots, x_d$ is said to be *good* if for all $i = 0, \ldots, t-1$ we have

$$D_i \cap (x_{d_i+1}, \ldots, x_d)M = 0,$$

where $d_i = \dim D_i$, $i = 0, \ldots, t-1$.

The dimension filtration $\mathcal{D}$ of M exists uniquely and $D_{t-1} = U_M(0)$. Moreover good systems of parameters of M always exist by [6, Lemma 2.5]. Notice that $0 :_M x_j = D_i$ for all $d_i + 1 \leq j \leq d_{i+1}$ by [5, Lemma 2.4]. We prove Theorem 1.2.

Proof of Theorem 1.2. By Corollary 2.6 we have $\cap_{n\geq 1}(x_1^n,\dots,x_j^n)_M^{\lim} \supseteq D_i$ for all $0\leq i\leq t-1$ and $d_i < j \leq d_{i+1}$. For every $m,n\geq 1$ we consider the submodule defined as follows

$$(x_1^n,\dots,x_j^n|x_{j+1}^m,\dots,x_d^m)_M^{\lim} := \bigcup_{k\geq 1}\big((x_1^{n(k+1)},\dots,x_j^{n(k+1)},x_{j+1}^m,\dots,x_d^m)M : (x_1...x_j)^{nk}\big).$$

We have

$$\frac{(x_1^n,\dots,x_j^n|x_{j+1}^m,\dots,x_d^m)_M^{\lim}}{(x_{j+1}^m,\dots,x_d^m)M} = (x_1^n,\dots,x_j^n)_{M/(x_{j+1}^m,\dots,x_d^m)M}^{\lim}.$$

Therefore by Theorem 1.1 we have

$$\bigcap_{n\geq 1}\frac{(x_1^n,\dots,x_j^n|x_{j+1}^m,\dots,x_d^m)_M^{\lim}}{(x_{j+1}^m,\dots,x_d^m)M} = U_{M/(x_{j+1}^m,\dots,x_d^m)M}(0).$$

So

$$\bigcap_{n\geq 1}(x_1^n,\dots,x_j^n|x_{j+1}^m,\dots,x_d^m)_M^{\lim} = \bigcap_{\mathfrak{p}\in \mathrm{Assh}M/(x_{j+1}^m,\dots,x_d^m)M} N_m(\mathfrak{p}), \quad (\star)$$

where $N_m(\mathfrak{p})$ is the $\mathfrak{p}$-primary component which appears in a reduced primary decomposition of $(x_{j+1}^m,\dots,x_d^m)M$.

Now suppose $D_i \subsetneq \cap_{n\geq 1}(x_1^n,\dots,x_j^n)_M^{\lim}$. Thus there exists $a\in \cap_{n\geq 1}(x_1^n,\dots,x_j^n)_M^{\lim}$ and $\dim Ra = d_s$ for some $s>i$. Hence $a\in D_s$ and there exists $\mathfrak{q}_0\in \mathrm{Assh}(Ra)$, that is $\dim R/\mathfrak{q}_0 = d_s$. Since $\underline{x}$ is a good system of parameters we have

$$D_s\cap (x_{d_s+1}^m,\dots,x_d^m)M = 0$$

for all $m\geq 1$. So

$$D_s \cong \frac{D_s + (x_{d_s+1}^m,\dots,x_d^m)M}{(x_{d_s+1}^m,\dots,x_d^m)M},$$

and consider the right hand side as a submodule of $M/(x_{d_s+1}^m,\dots,x_d^m)M$. Therefore $\mathfrak{q}_0\in \mathrm{Assh}M/(x_{d_s+1}^m,\dots,x_d^m)M$ for all $m\geq 1$. Because $\dim R/\mathfrak{q}_0 = d_s$, $x_1,\dots,x_s$ is a system of parameters of $R/\mathfrak{q}_0$. We have $\dim R/(\mathfrak{q}_0 + (x_{j+1},\dots,x_{d_s})) = j$. Choose a prime ideal $\mathfrak{p}_0\in \mathrm{Assh}R/(\mathfrak{q}_0+(x_{j+1},\dots,x_{d_s}))$, so $\dim R/\mathfrak{p}_0 = j$. For each $m\geq 1$, since $\mathfrak{q}_0\in \mathrm{Assh}M/(x_{d_s+1}^m,\dots,x_d^m)M$ we have $\mathrm{Ann}(M/(x_{d_s+1}^m,\dots,x_d^m)M)\subseteq \mathfrak{q}_0$. So

$$\mathrm{Ann}(M/(x_{j+1}^m,\dots,x_d^m)M)\subseteq \sqrt{\mathfrak{q}_0+(x_{j+1},\dots,x_{d_s})}\subseteq \mathfrak{p}_0.$$

Therefore $\mathfrak{p}_0\in \mathrm{Supp}(M/(x_{j+1}^m,\dots,x_d^m)M))$. However $\dim R/\mathfrak{p}_0 = \dim M/(x_{j+1}^m,\dots,x_d^m)M) = j$. Thus we have

$$\mathfrak{p}_0\in \mathrm{Assh}M/(x_{j+1}^m,\dots,x_d^m)M$$

for all $m\geq 1$. Since $a\in \cap_{n\geq 1}(x_1^n,\dots,x_j^n)_M^{\lim}$ we have

$$a\in \cap_{n\geq 1}(x_1^n,\dots,x_j^n|x_{j+1}^m,\dots,x_d^m)_M^{\lim}$$

for all $m\geq 1$. By $(\star)$ we have $a\in \cap_{m\geq 1}N_m(\mathfrak{p}_0)$. Localization at $\mathfrak{p}_0$ we have $\frac{a}{1}\neq 0\in M_{\mathfrak{p}_0}$ since $\mathfrak{q}_0\in \mathrm{Ass}(Ra)$ and $\mathfrak{q}_0\subseteq \mathfrak{p}_0$. So $\cap_{m\geq 1}(N_m(\mathfrak{p}_0))_{\mathfrak{p}_0}\neq 0$. On the other hand, since

$\mathfrak{p}_0 \in \mathrm{Assh} M/(x_{j+1}^m, \ldots, x_d^m)M$, it is a minimal associated prime of $M/(x_{j+1}^m, \ldots, x_d^m)M$. Therefore

$$(N_m(\mathfrak{p}_0))_{\mathfrak{p}_0} = (x_{j+1}^m, \ldots, x_d^m)M_{p_0}.$$

Hence $\cap_{m\geq 1}(N_m(\mathfrak{p}_0))_{\mathfrak{p}_0} = 0$ by the Krull intersection theorem. This is a contradiction. The proof is complete. $\square$

The following is a generalization of Corollary 3.2.

Corollary 4.2. *Let M be a finitely generated R-module of dimension d with the dimension filtration $\mathcal{D} : H_{\mathfrak{m}}^0(M) = D_0 \subseteq D_1 \cdots \subseteq D_{t-1} \subseteq D_t = M$, where $\dim D_i = d_i$ for all $i = 0, \ldots t$. For all $0 \leq i \leq t-1$ and $d_i < j \leq d_{i+1}$ we have $\cap_{\underline{x}}(\underline{x})_M^{\lim} = D_i$, where $\underline{x} = x_1, \ldots, x_j$ runs through all part of systems of parameters of M.*

Remark 4.3. The proof of Theorem 1.2 will be much more easy if we assume $\underline{x} = x_1, \ldots, x_d$ is a dd-sequence of M (for the definition of dd-sequences see [6]). It is known that if a system of parameters is a dd-sequence, then it is a good system of parameters (cf. [5, Corollary 3.7]). Furthermore, every finitely generated R-module admits a system of parameter which is a dd-sequence if and only if the ring is an image of a Cohen-Macaulay local ring (cf. [7]). For each $m \geq 1$ we have $x_1, \ldots, x_j$ is a dd-sequence of $M/(x_{j+1}^m, \ldots, x_d^m)M$, so

$$U_{M/(x_{j+1}^m,\ldots,x_d^m)M}(0) = \frac{(x_{j+1}^m, \ldots, x_d^m)M :_M x_j}{(x_{j+1}^m, \ldots, x_d^m)M}$$

for all $m \geq 1$. Following the proof of Theorem 1.2, for all $m \geq 1$ we have

$$\bigcap_{n\geq 1}(x_1^n, \ldots, x_j^n | x_{j+1}^m, \ldots, x_d^m)_M^{\lim} = (x_{j+1}^m, \ldots, x_d^m)M :_M x_j.$$

Therefore

$$\begin{aligned}
\bigcap_{n\geq 1}(x_1^n, \ldots, x_j^n)_M^{\lim} &\subseteq \bigcap_{n,m\geq 1}(x_1^n, \ldots, x_j^n | x_{j+1}^m, \ldots, x_d^m)_M^{\lim} \\
&= \bigcap_{m\geq 1}\bigcap_{n\geq 1}(x_1^n, \ldots, x_j^n | x_{j+1}^m, \ldots, x_d^m)_M^{\lim} \\
&= \bigcap_{m\geq 1}((x_{j+1}^m, \ldots, x_d^m)M :_M x_j) \\
&= 0 :_M x_j = D_i
\end{aligned}$$

the last equation follows from [5, Lemma 2.4].

5. Systems of parameters

In this section, we prove a characterization of systems of parameters in terms of injectivity of the determinatal maps. Our results improve known results of Dutta and Roberts in [12] and of Fouli and Huneke in [13]. Let $\underline{x} = x_1, \ldots, x_t$ be a system of parameters of R, and $\underline{y} = y_1, \ldots, y_t$ a sequence of elements such that $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$. Dutta and Roberts proved in [12] that if R is a Cohen-Macaulay ring, then $\underline{y}$ is a system of parameters if and only if the determinantal map $\det A : R/(\underline{x}) \to R/(\underline{y})$ is injective. The following result is a slight generalization of Dutta-Roberts's result.

Theorem 5.1. *Let R be a local ring such that $R/U_R(0)$ is Cohen-Macaulay. Let $\underline{x}$ be a system of parameters and $\underline{y}$ a sequence of elements in R such that $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$. Then $\underline{y}$ is a system of parameters of $R/U_R(0)$ if and only if the determinantal map $\det A : R/(\underline{x})^{\lim} \to R/(\underline{y})^{\lim}$ is injective.*

Proof. By Corollary 2.6 we have that $U_R(0) \subseteq (\underline{x}^{[s]})^{\lim}$ for all $s \geq 1$. Set $R' = R/U_R(0)$. For an element $a \in R$, we denote by a' the image of a in R'. Set $\underline{x}' = x'_1, \ldots, x'_t$ and $\underline{y}' = y'_1, \ldots, y'_t$. We also have $(\underline{y}') \overset{A}{\subseteq} (\underline{x}')$, here we consider R' as an R-module. By Proposition 2.7 we have

$$(\underline{x}')^{\lim}_{R'} = (\underline{x})^{\lim}/U_R(0); (\underline{y}')^{\lim}_{R'} = (\underline{y})^{\lim}/U_R(0).$$

Therefore both deteminatal maps

$$\det A : R/(\underline{x})^{\lim} \to R/(\underline{y})^{\lim}$$

and

$$\det A : R'/(\underline{x}')^{\lim}_{R'} \to R'/(\underline{y}')^{\lim}_{R'}$$

are the same. Notice that R' is Cohen-Macaulay. Then the conclusion follows from the module-version of mentioned above result of Dutta and Roberts. □

As a consequence of Theorem 5.1 we obtain a recently result of Fouli and Huneke [13, Theorem 4.4] as follows.

Corollary 5.2. *Let R be a 1-dimensional Noetherian local ring. Let x be a parameter, and let $y = ux$. Then y is a parameter if and only if the map $R/(x)^{\lim} \xrightarrow{u} R/(y)^{\lim}$ is injective.*

Proof. Since $\dim R = 1$ we have that $U_R(0) = H^0_{\mathfrak{m}}(R)$ and $\overline{R} = R/U_R(0)$ is Cohen-Macaulay. Moreover, x is a parameter of R if and only if $\overline{x}$ is also a parameter of $\overline{R}$. Hence the assertion follows from Theorem 5.1. □

In higher dimension, we need the condition that R is equidimensional to claim that if a sequence $\underline{x} = x_1, ..., x_t$ is a system of parameters of $R/U_R(0)$, then it is a system of parameters of R. We need the following in the sequel.

Lemma 5.3. *Let $(R, \mathfrak{m})$ be a catenary equidimensional local ring of dimension t, $\underline{x} = x_1, ..., x_t$ a sequence of elements of R. Then the following statements are equivalent*

(i) *$\underline{x}$ is a system of parameters of R.*

(ii) *$\underline{x}$ is a system of parameters of $\widehat{R}/U_{\widehat{R}}(0)$.*

Proof. (i) ⇒ (ii) is clear.

(ii) ⇒ (i) It is easily seen that $\underline{x}$ is a system of parameters of $\widehat{R}/U_{\widehat{R}}(0)$ if and only if $\underline{x}$ is a system of parameters of $\widehat{R}/\mathfrak{P}$ for all $\mathfrak{P} \in \mathrm{Assh}\widehat{R}$. And hence we shall prove by induction on t that if $\underline{x}$ is a system of parameters of $\widehat{R}/\mathfrak{P}$ for all $\mathfrak{P} \in \mathrm{Assh}\widehat{R}$, then $\underline{x}$ is a system of parameters of R. The case $t = 1$ is trivial. Suppose that $t > 1$. We choose an arbitrary prime ideal $\mathfrak{p} \in \mathrm{Assh}R$, then there exists $\mathfrak{P} \in \mathrm{Assh}\widehat{R}$ such that $\mathfrak{P} \cap R = \mathfrak{p}$. Since $\underline{x}$ is a system of parameters of $\widehat{R}/\mathfrak{P}$ we have $x_1 \notin \mathfrak{P}$, and so $x_1 \notin \mathfrak{p}$. Thus $R/(x_1)$ is a catenary equidimensional local ring of dimension $t-1$. We shall show that $\underline{x}' = x_2, ..., x_t$ is a system of parameter of $\widehat{R}/\mathfrak{P}'$ for all $\mathfrak{P}' \in \mathrm{Assh}\widehat{R}/(x_1)\widehat{R}$. If $\mathfrak{P}' \in \min(\widehat{R})$, then $\mathfrak{P}' \cap R = \mathfrak{p} \in \mathrm{Assh}R$ since R is equdimensional. Therefore $x_1 \notin \mathfrak{P}'$, it is a contradiction. Thus there exists $\mathfrak{P} \in \mathrm{Assh}\widehat{R}$ such that $\mathfrak{P} \subsetneq \mathfrak{P}'$, note that $\dim \widehat{R}/\mathfrak{P}' = d-1$. Because $\underline{x} = x_1, \ldots, x_t$ is a system of parameters of $\widehat{R}/\mathfrak{P}$ we have $\underline{x}' = x_2, ..., x_t$ is a system of parameter of $\widehat{R}/\mathfrak{P}'$. By the inductive hypothesis $\underline{x}' = x_2, ..., x_t$ is a system of parameters of $R/(x_1)$. Thus $\underline{x}$ is a system of parameters of R as required. $\square$

The following is the main result of this section.

Theorem 5.4. *Let $(R, \mathfrak{m})$ be a catenary local ring of dimension t. There exists a positive integer ℓ, which depends only on R, with property: whenever $\underline{x} = x_1, ..., x_t$ is a system of parameters of $R/U_R(0)$ with $(\underline{x}) \subseteq \mathfrak{m}^\ell$ and $\underline{y} = y_1, \ldots, y_t$ a sequence of elements such that $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$ the following statements are equivalent*

(i) *$\underline{y}$ forms a system of parameters of $R/U_R(0)$.*

(ii) *The determinantal map $R/(\underline{x})^{\lim} \overset{\det A}{\longrightarrow} R/(\underline{y})^{\lim}$ is injective.*

Proof. (i) $\Rightarrow$ (ii) follows from Remark 2.3 and the same argument as the proof of Theorem 5.1.
(ii) $\Rightarrow$ (i). We first show that the assertion in the case R is complete. By Corollary 2.6 and Proposition 2.7 we may assume henceforth that $U_R(0) = 0$ and hence R is equidimensional. Assume that $\mathrm{Ass}R = \mathrm{Assh}R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $\cap_{\mathfrak{p}_i \in \mathrm{Ass}\, R} N(\mathfrak{p}_i) = 0$ is a reduced primary decomposition of (0). For $1 \le i \le n$ we set

$$L_i = \bigcap_{j \neq i, \mathfrak{p}_j \in \mathrm{Ass}R} N(\mathfrak{p}_j).$$

Let $\underline{z} = z_1, ..., z_t$ is a system of parameters of R. By Theorem 1.1 we have $\cap_{n \ge 1} (\underline{z}^{[n]})^{\lim} = 0$. Then there is a positive integer ℓ_1 such that $N_i \nsubseteq (\underline{z}^{[\ell_1]})^{\lim}$ for all $i = 1, \ldots, n$. Let ℓ be a positive integer such that $\mathfrak{m}^\ell \subseteq (\underline{z}^{[\ell_1]})$.

Suppose we have $\underline{x} = x_1, ..., x_t$ and $\underline{y} = y_1, \ldots, y_t$ are sequences of elements contained in $\mathfrak{m}^\ell$ such that $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$ and $\underline{x}$ is a system of parameters of R and the determinantal map $R/(\underline{x})^{\lim} \overset{\det A}{\longrightarrow} R/(\underline{y})^{\lim}$ is injective. Assume $\underline{y}$ is not a system of parameter of R. By relabeling (if necessarily) we can assume henceforth that $\underline{y} = y_1, \ldots, y_t$ is not a system of parameters of $R/\mathfrak{p}_1$. By Corollary 3.5 we have $0 \neq L_1 \subseteq (\underline{y})^{\lim}$. On the other hand, it follows from Remark 2.3 that $(\underline{x})^{\lim} \subseteq (\underline{z}^{[\ell_1]})^{\lim}$. Hence $L_i \nsubseteq (\underline{x})^{\lim}$ for all $i = 1, \ldots, n$. Thus there is $0 \neq u \in (\underline{y})^{\lim} \setminus (\underline{x})^{\lim}$. Therefore the determinantal map $R/(\underline{x})^{\lim} \overset{\det A}{\longrightarrow} R/(\underline{y})^{\lim}$ maps $u + (\underline{x})^{\lim}$ to 0. So it is not injective. This is a contradiction. Hence $\underline{y} = y_1, \ldots, y_t$ is a system of parameters of R.
The assertion in general case now follows from Lemma 5.3 since $R/U_R(0)$ is catenary and equidimensional. The proof is complete. $\square$

Proof of Theorem 1.3. It immediately follows from Theorem 5.4 and the fact $\underline{x}$ is a system of parameters of R if and only if $\underline{x}$ is a system of parameters of $R/U_R(0)$ provided R is equidimensional. □

Theorem 1.3 was proved by Fouli and Huneke for any equidimensional local ring (cf. [13, Corollary 5.4]). However, in fact, they proved this result with assumption that R is complete. Thus our result is a generalization of their one. We will show that the catenary condition is essential.

Example 5.5 (see, [25], Example 2, pp 203–205). Let K be a field, and $K[[X]]$ a formal power series ring. Let $Z = \sum_{i\geq 1} a_i x^i$ be a algebraically independent element over $K(X)$. Set $Z_j = (Z - \sum_{k<j} a_k X^k)/X^{j-1}$. Furthermore let Y be a algebraically independent element over $K[X, Z]$. Let $R_1 = K[X, Z_1, \ldots, Z_j, ...]$ and set $R_2 = R_1[Y]$. Let $\mathfrak{n}_1 = (X, Y)$, $\mathfrak{n}_2 = (X - 1, Z, Y)$ are maximal ideals of R_2 with $\mathrm{ht}(\mathfrak{n}_1) = 2$ and $\mathrm{ht}(\mathfrak{n}_2) = 3$. Let S be the intersection of complements of $\mathfrak{n}_1$ and $\mathfrak{n}_2$ in R_2 and set $R' = (R_2)_S$. Then R' is Noetherian. Let $\mathfrak{m}$ be the Jacobson radical of R' and set $R = K + \mathfrak{m}$. We have $(R, \mathfrak{m})$ is a local domain of dimension 3. However R is non-catenary since $0 \subset \mathfrak{q} = XR' \cap R \subset \mathfrak{m}$ is a maximal chain of prime ideals in R. Since $(R, \mathfrak{m})$ is a local ring of dimension three, $\mathfrak{q}$ is generated by three element up to radical.

Proposition 5.6. *Let $R, \mathfrak{m}, \mathfrak{q}$ as in the previous Example, and let $\underline{x} = x_1, x_2, x_3$ be any system of parameters of R. Then there exists $\underline{y} = y_1, y_2, y_3$ such that $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$ and the determinantal map $R/(\underline{x})^{\lim} \xrightarrow{\det A} R/(\underline{y})^{\lim}$ is injective but $\underline{y}$ is not a system of parameters.*

Proof. As above, $\mathfrak{q} = \sqrt{(z_1, z_2, z_3)}$ for some z_1, z_2, z_3. We may choose the sequence $\underline{y} = y_1, y_2, y_3$ with $y_i = z_i^k$, $1 \leq i \leq 3$, for large enough k such that $(\underline{y}) \overset{A}{\subseteq} (\underline{x})$. It is clear that $\underline{y}$ is not a system of parameters of R. We now show that the determinantal map

$$R/(\underline{x})^{\lim} \xrightarrow{\det A} R/(\underline{y})^{\lim}$$

is injective. It is equivalent to prove that the determinantal map

$$\widehat{R}/(\underline{x})^{\lim}_{\widehat{R}} \xrightarrow{\det A} \widehat{R}/(\underline{y})^{\lim}_{\widehat{R}}$$

is injective By Theorem 5.4 it is sufficient to prove that $\underline{y}$ is a system of parameters of $\widehat{R}/\mathfrak{P}$ for any $\mathfrak{P} \in \mathrm{Assh}\widehat{R}$, that is $\dim \widehat{R}/\mathfrak{P} = 3$. Indeed, let $\mathfrak{q}\widehat{R} = \mathfrak{Q}_1 \cap \cdots \cap \mathfrak{Q}_r$, where $\mathfrak{Q}_i$ is a $\mathfrak{P}_i$-primary, is a reduced primary decomposition of $\mathfrak{q}\widehat{R}$. Since $\dim R/\mathfrak{q} = 1$ we have $\dim \widehat{R}/\mathfrak{P}_i = 1$ and $\mathrm{ht}(\mathfrak{P}_i/\mathfrak{q}\widehat{R}) = 0$ for all $i \leq r$. Thus by [23, Theorem 15.1] we have $\mathrm{ht}(\mathfrak{P}_i) = \mathrm{ht}(\mathfrak{q}) + \mathrm{ht}(\mathfrak{P}_i/\mathfrak{q}\widehat{R}) = 1$ for all $i \leq r$. Moreover, $\widehat{R}$ is catenary, so $\mathfrak{P} \nsubseteq \mathfrak{P}_i$ for all $i \leq r$. Therefore $\dim \widehat{R}/(\mathfrak{q}\widehat{R} + \mathfrak{P}) = 0$. Hence $\underline{y}$ is a system of parameters of $\widehat{R}/\mathfrak{P}$ for all $\mathfrak{P} \in \mathrm{Assh}\widehat{R}$ since $\sqrt{(\underline{y})} = \mathfrak{q}$. The proof is complete. □

6. A characterization of unmixed local rings

Unmixed local rings were introduced first by Nagata [25] as follows.

Definition 6.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension t. Then R is *unmixed* if $U_{\widehat{R}}(0) = 0$ i.e. $\operatorname{Assh} \widehat{R} = \operatorname{Ass} \widehat{R}$, where $\widehat{R}$ denotes the completion of R with respect to the $\mathfrak{m}$-adic topology.

Almost of domains in Commutative Algebra are unmixed. However, in [25, Example 2, pp. 203–205] Nagata constructed a domain of dimension two which is not unmixed. Unmixed local rings were investigated by several authors (cf. [27, 28, 31]). Let $\underline{x} = x_1, \ldots, x_t$ be a system of parameters of R. By Krull's intersection theorem we have $\cap_{n \geq 1}(\underline{x}^{[n]}) = 0$. It means that the topology defined by $\{(\underline{x}^{[n]})\}_{n \geq 1}$ is always Hausdorff. However, the topology defined by $\{(\underline{x}^{[n]})^{\lim}\}_{n \geq 1}$ may be not Hausdorff. In fact, $\{(\underline{x}^{[n]})^{\lim}\}_{n \geq 1}$ is a Hausdorff topology if and only if $U_R(0) = 0$ by Theorem 1.1. The aim of this section is to give a characterization of unmixed local rings in terms of the topology defined by $\{(\underline{x}^{[n]})^{\lim}\}_{n \geq 1}$.

Lemma 6.2. *Let $\underline{x} = x_1, \ldots, x_t$ and $\underline{y} = y_1, \ldots, y_t$ be systems of parameters of R. Then the topology defined by $\{(\underline{x}^{[n]})\}_{n \geq 1}$ and $\{(\underline{y}^{[n]})\}_{n \geq 1}$ are equivalent.*

Proof. For each $n \geq 1$ there is an integral $v(n)$ such that $(\underline{x}^{[n]}) \supseteq (\underline{y}^{[v(n)]})$. By Remark 2.3 we have $(\underline{x}^{[n]})^{\lim} \supseteq (\underline{y}^{[v(n)]})^{\lim}$. Therefore the topology defined by $\{(\underline{y}^{[n]})^{\lim}\}_{n \geq 1}$ is stronger or equal to the topology defined by $\{(\underline{x}^{[n]})^{\lim}\}_{n \geq 1}$. Symmetrically, we have the converse. The proof is complete. $\square$

By the previous Lemma we can define a topology of R as follows.

Definition 6.3. Let $(R, \mathfrak{m})$ be a local ring of dimension t. We define the *limit closure topology* of R the topology defined by $\{(\underline{x}^{[n]})\}_{n \geq 1}$ for some system of parameters $\underline{x} = x_1, \ldots, x_t$.

The following result, proved by Chevalley (cf. [3, Lemma 7]), plays the key role in our proof the main result of this section.

Lemma 6.4 (Chevalley)**.** *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring, and $\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$ a chain of ideals of R such that $\cap_{n \geq 1} \mathfrak{a}_n = 0$. Then for each n there exists an integer $v(n)$ such that $\mathfrak{a}_{v(n)} \subseteq \mathfrak{m}^n$. In other words, the linear topology defined by $\{\mathfrak{a}_n\}_{n \geq 1}$ is stronger or equal to the $\mathfrak{m}$-adic topology.*

We now prove Theorem 1.4 proposed in the introduction.

Theorem 6.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension t. Then R is unmixed if and only if the $\mathfrak{m}$-adic and limit closure topologies are equivalent.*

Proof. We note that the $\mathfrak{m}$-adic topology is always stronger or equal to the topology defined by $\{(\underline{x}^{[n]})^{\lim}\}_{n \geq 1}$ since $(\underline{x}^{[n]})^{\lim}$ is $\mathfrak{m}$-primary for all $n \geq 1$.
($\Rightarrow$) We assume that R is unmixed. Then by Theorem 1.1 the topology defined by $\{(\underline{x}^{[n]})^{\lim}_{\widehat{R}}\}_{n \geq 1}$ is Hausdorff. By Chevalley's theorem, for each n there exists an integer $v(n)$ such that $(\underline{x}^{[v(n)]})^{\lim}_{\widehat{R}} \subseteq \widehat{\mathfrak{m}}^n$. Thus $(\underline{x}^{[v(n)]})^{\lim} \subseteq \mathfrak{m}^n$. Therefore the topology defined by $\{(\underline{x}^{[n]})^{\lim}\}_{n \geq 1}$ is stronger or equal to the $\mathfrak{m}$-adic topology. So they are equivalence.
($\Leftarrow$) Suppose that R is not unmixed i.e. $U_{\widehat{R}}(0) \neq 0$. By Krull's intersection theorem, there exists n_0 such that $U_{\widehat{R}}(0) \nsubseteq \widehat{\mathfrak{m}}^{n_0}$. On the other hand, we get by Theorem 1.1 that $U_{\widehat{R}}(0) \subseteq (\underline{x}^{[n]})^{\lim}_{\widehat{R}}$ for all $n \geq 1$. Therefore $(\underline{x}^{[n]})^{\lim}_{\widehat{R}} \nsubseteq \widehat{\mathfrak{m}}^{n_0}$ for all n. Thus $(\underline{x}^{[n]})^{\lim} \nsubseteq \mathfrak{m}^{n_0}$ for all $n \geq 1$ so the topology defined by $\{(\underline{x}^{[n]})^{\lim}\}_{n \geq 1}$ is not equivalent to the $\mathfrak{m}$-adic topology. $\square$

Corollary 6.6. *Let $(R,\mathfrak{m})$ be a Noetherian local ring such that $U_R(0)=0$. Suppose that the $\mathfrak{m}$-adic topology is minimal among all Hausdorff topologies of R. Then R is unmixed.*

Example 6.7. By Nagata (cf. [25, Example 2, pp. 203–205]) we have there exists a local domain $(R,\mathfrak{m})$ of dimension two such that $\widehat{R}\cong k[[X,Y,Z]]/((X)\cap(Y,Z))=k[[x,y,z]]$, where k be a field. Let a,b be a system of parameters of R. We have $U_{\widehat{R}}(0)=(x)\notin\widehat{\mathfrak{m}}^2$. Therefore, $(a^n,b^n)^{\lim}_{\widehat{R}}\nsubseteq\widehat{\mathfrak{m}}^k$ for every $k\geq 2$. Thus $\{(a^n,b^n)^{\lim}\}_{n\geq 1}$ is a Hausdorff topology of R by $(a^n,b^n)^{\lim}\nsubseteq\mathfrak{m}^k$ for all $n\geq 1$ and all $k\geq 2$.

7. Limit closure in local rings of dimension two

It is easy to see that if $\dim R=2$ then the monomial conjecture holds true. So $(\underline{x})^{\lim}\subseteq\mathfrak{m}$. In his breakthrough paper [16] Heitmann extended it for any local ring of dimension at most three. The purpose of the section is to give some explicit descriptions for the limit closure in local rings of dimension two. Let $\underline{x}=x_1,\ldots,x_d$ be a system of parameter of the local ring $(R,\mathfrak{m})$. Since $(\underline{x})^{\lim}_{\widehat{R}}=(\underline{x})^{\lim}\widehat{R}$, we shall assume that $(R,\mathfrak{m})$ is an image of a Cohen-Macaulay local ring.

S_2-ification.(cf. [19]) Suppose R is an unmixed local ring. We shall say that a ring S is an *S_2-ification* of R if it lies between R and its total quotient ring, is module-finite over R, is S_2 as an R-module, and has the property that for every element $s\in S-R$, the ideal $D(s)$, defined as $\{r\in R|rs\in R\}$, has height at least two.

Remark 7.1. (i) If $(R,\mathfrak{m})$ is complete, R has a S_2-ification, and it is unique. Moreover, if ω is a canonical module of R, then $S\cong\mathrm{Hom}(\omega,\omega)$.

(ii) Let $\mathfrak{a}_i=\mathrm{Ann}H^i_{\mathfrak{m}}(R)$, $i=1,\ldots,d$, and $\mathfrak{a}=\mathfrak{a}_0...\mathfrak{a}_{d-1}$. If R is an image of a Cohen-Macaulay local ring, we have $\dim R/\mathfrak{a}_i\leq i$ and so $\dim R/\mathfrak{a}\leq d-1$ (cf. [2, Theorem 8.1.1], [7]). Moreover if R is unmixed we have $\dim R/\mathfrak{a}\leq d-2$. We have the S_2-ification of R is just the ideal transformation $D_{\mathfrak{a}}(R)$. Thus the S_2-ification of an unmixed local ring exists provided the ring is an image of a Cohen-Macaulay local ring.

The following is implicit in the proof of [24, Theorem 4.3]. For the sake of completeness we give a detail proof.

Theorem 7.2. *Let $(R,\mathfrak{m})$ be an unmixed local ring of dimension d and $\underline{x}=x_1,\ldots,x_d$ a system of parameters of R. Let S is the S_2-ification of R. Then $(\underline{x})^{\lim}=(\underline{x})^{\lim}_S\cap R$.*

Proof. Consider the exact sequence

$$0\to R\to S\to S/R\to 0,$$

with $\dim S/R \le d-2$. Applying the functors local cohomology and Koszul's cohomology we have the following commutative diagram.

$$\begin{tikzcd}
 & & H^{d-1}(\underline{x};S/R) \arrow[r] \arrow[d] & H^{d-1}_{\mathfrak{m}}(S/R)=0 \arrow[d] \\
0 \arrow[r] & (\underline{x})^{\lim}/(\underline{x})R \arrow[r] \arrow[d] & H^d(\underline{x};R) \arrow[r] \arrow[d] & H^d_{\mathfrak{m}}(R) \arrow[d] \\
0 \arrow[r] & (\underline{x})^{\lim}_S/(\underline{x})S \arrow[r] & H^d(\underline{x};S) \arrow[r] \arrow[d] & H^d_{\mathfrak{m}}(S) \arrow[d] \\
 & & H^d(\underline{x};S/R) \arrow[r] & H^d_{\mathfrak{m}}(S/R)=0.
\end{tikzcd}$$

Both the second and third rows are exact by Remark 2.2. Thus we have the following commutative diagram

$$\begin{tikzcd}
R/(\underline{x})^{\lim} \arrow[r, "\pi"] \arrow[d, "\alpha"'] & H^d_{\mathfrak{m}}(R) \arrow[d, "\sigma"'] \\
S/(\underline{x})^{\lim}_S \arrow[r, "\tau"] & H^d_{\mathfrak{m}}(S)
\end{tikzcd}$$

with π and τ are injective and σ is bijective. The equation $\tau \circ \alpha = \sigma \circ \pi$ implies that the map

$$\alpha : R/(\underline{x})^{\lim} \to S/(\underline{x})^{\lim}_S$$

is injective. Therefore $(\underline{x})^{\lim} = (\underline{x})^{\lim}_S \cap R$. $\square$

In the rest of this section we assume that $\dim R = 2$ and x, y a system of parameters of R. Let $U_R(0)$ is the unmixed component of R and $\overline{R} = R/U_R(0)$. By Theorem 1.1 and Proposition 2.7 we have

$$(x,y)^{\lim} = \bigcup_{n\ge 1} \big((x^{n+1}, y^{n+1}, U_R(0)) :_R (xy)^n\big)$$

and $\ell(R/(x,y)^{\lim}) = \ell(\overline{R}/(x,y)^{\lim}_{\overline{R}})$. Hence we can reduce to the case R is unmixed (cf. [2, Theorem 2.1.15]). In this is the case we have the S_2-ification S of R is Cohen-Macaulay (since $d = 2$). Moreover, $H^1_{\mathfrak{m}}(R)$ is finitely generated (see, [31]) and $S/R \cong H^1_{\mathfrak{m}}(R)$. The following is the main result of this section.

Proof of Theorem 1.5. (i) follows from Theorem 7.2 and the Cohen-Macaulayness of S.
(ii) By (i) the short exact sequence

$$0 \to R \to S \to H^1_{\mathfrak{m}}(R) \to 0$$

induces the short exact sequence

$$0 \to R/(x,y)^{\lim} \to S/(x,y)S \to H^1_{\mathfrak{m}}(R)/(x,y)H^1_{\mathfrak{m}}(R) \to 0.$$

Therefore $\ell(R/(x,y)^{\lim}) = \ell(S/(x,y)S) - \ell(H^1_{\mathfrak{m}}(R)/(x,y)H^1_{\mathfrak{m}}(R))$. Since S is Cohen-Macaulay we have $\ell(S/(x,y)S) = e(x,y;S) = e(x,y;R)$. Thus we get the assertion.

(iii) We first claim that $\ell(R/(x,y)) = e(x,y;R) + \ell(0 :_{H^1_{\mathfrak{m}}(R)} (x,y))$. Indeed, let $R' = R/(x)$. From the short exact sequence

$$0 \to R \xrightarrow{x} R \to R' \to 0$$

we have the exact sequence of local cohomology

$$0 \to H^0_{\mathfrak{m}}(R') \to H^1_{\mathfrak{m}}(R) \xrightarrow{x} H^1_{\mathfrak{m}}(R) \to \cdots .$$

So $H^0_{\mathfrak{m}}(R') \cong 0 :_{H^1_{\mathfrak{m}}(R)} x$. Since x is a regular element we have $e(x,y;R) = e(y;R')$. Notice that $\dim R' = 1$, we can check that $H^0_{\mathfrak{m}}(R') \cap (y)R' = yH^0_{\mathfrak{m}}(R')$. Thus we have the short exact sequence

$$0 \to H^0_{\mathfrak{m}}(R')/yH^0_{\mathfrak{m}}(R') \to R'/(y)R' \to \overline{R}/(y)\overline{R} \to 0,$$

where $\overline{R} = R'/H^0_{\mathfrak{m}}(R')$. Since $\overline{R}$ is Cohen-Macaulay we have

$$\ell(\overline{R}/(y)\overline{R}) = e(y;\overline{R}) = e(y;R') = e(x,y;R).$$

Therefore, following the above short exact sequence we have

$$\ell(R/(x,y)) = \ell(R'/(y)R') = e(x,y;R) + \ell(H^0_{\mathfrak{m}}(R')/yH^0_{\mathfrak{m}}(R')).$$

Consider the following exact sequence of finite length modules

$$0 \to 0 :_{H^0_{\mathfrak{m}}(R')} y \to H^0_{\mathfrak{m}}(R') \xrightarrow{y} H^0_{\mathfrak{m}}(R') \to H^0_{\mathfrak{m}}(R')/yH^0_{\mathfrak{m}}(R') \to 0.$$

It follows that $\ell(H^0_{\mathfrak{m}}(R')/yH^0_{\mathfrak{m}}(R')) = \ell(0 :_{H^0_{\mathfrak{m}}(R')} y)$. On the other hand, since $H^0_{\mathfrak{m}}(R') \cong 0 :_{H^1_{\mathfrak{m}}(R)} x$ we have $0 :_{H^0_{\mathfrak{m}}(R')} y \cong 0 :_{H^1_{\mathfrak{m}}(R)} (x,y)$. Thus we have

$$\ell(R/(x,y)) = e(x,y;R) + \ell(0 :_{H^1_{\mathfrak{m}}(R)} (x,y)).$$

Combining the above assertion with (ii) we have

$$\ell((x,y)^{\lim}/(x,y)) = \ell(0 :_{H^1_{\mathfrak{m}}(R)} (x,y)) + \ell(H^1_{\mathfrak{m}}(R)/(x,y)H^1_{\mathfrak{m}}(R)).$$

Notice that $0 :_{H^1_{\mathfrak{m}}(R)} (x,y) \cong H^0(x,y;H^1_{\mathfrak{m}}(R))$ and $H^1_{\mathfrak{m}}(R)/(x,y)H^1_{\mathfrak{m}}(R) \cong H^2(x,y;H^1_{\mathfrak{m}}(R))$. We have the Euler charactiristic

$$\chi(x,y;H^1_{\mathfrak{m}}(R)) = \sum_{i=0}^{2} \ell(H^i(x,y;H^1_{\mathfrak{m}}(R))) = 0$$

since $\dim H^1_{\mathfrak{m}}(R) < 2$ (cf. [2, Theorem 4.7.6]). Therefore

$$\ell((x,y)^{\lim}/(x,y)) = \ell(H^1(x,y;H^1_{\mathfrak{m}}(R))).$$

The proof is complete. $\square$

Corollary 7.3. *Let $(R,\mathfrak{m})$ be an equidimensional local ring of dimension two, which is an image of a Cohen-Macaulay local ring. Let x,y be a system of parameters of R. Then $(x,y)^{\lim} \subseteq \overline{(x,y)}$, the integral closure of (x,y).*

Proof. It is easy to reduce to the case R is unmixed. Let S is the S_2-ification of R we have $(x,y)^{\lim} = (x,y)S \cap R$. Since S is a finite extension of R, the assertion follows from [22, Proposition 1.6.1]. $\square$

8. Some Examples

The main aim of this section is to compute a certain limit closure. The following example is based on [21, Example 6.2].

Example 8.1. Let K be a field of characteristic 0 and let $A = K[X, Y, U, V]/(f)$ where $f = XY - UX^2 - VY^2$. We denote by x, y, u, v the images of X, Y, U, V, respectively. Set $R = A_{\mathfrak{m}}$ where $\mathfrak{m} = (x, y, u, v)$ and $\mathfrak{p} = (y, u, v)$. It is easy to prove that R is a Gorenstein three-dimensional domain, and $\mathfrak{p}$ is a height two prime ideal of R. After completion one can prove that f factors into two formal series $f = (x - vy + \cdots)(y - ux + \cdots)$, where every term in the element $x - vy + \cdots$ lies in (y, u, v) except for the first term x (we can prove this fact by induction). We shall see these factors more detail in Proposition 8.5. There are two minimal primes lying over (0) in $\widehat{R}$, and $\mathfrak{p}\widehat{R} + (x - vy + \cdots) = \widehat{\mathfrak{m}}$. The Lichtenbaum-Hartshore vanishing theorem implies that $H^3_{\mathfrak{p}}(R) \neq 0$. R is a domain, by Proposition 2.8 we have $\operatorname{Ann} H^3_{\mathfrak{p}}(R) = 0$, so $\cap_{n\geq 1}(y^n, u^n, v^n)^{\lim} = 0$. However the Hausdorff topology defined by $\{(y^n, u^n, v^n)^{\lim}\}_{n\geq 1}$ is not equivalent to the $\mathfrak{m}$-adic topology. Indeed, by Lemma 2.5, $(y^n, u^n, v^n)^{\lim}$ is $\mathfrak{m}$-primary for all n. Hence the $\mathfrak{m}$-adic topology is stronger or equal to the $\{(y^n, u^n, v^n)^{\lim}\}_{n\geq 1}$ topology. By Corollary 3.5 we have $(x - vy + \cdots) \subseteq (y^n, u^n, v^n)^{\lim}_{\widehat{R}}$ for all $n \geq 1$. Thus $(y^n, u^n, v^n)^{\lim}_{\widehat{R}} \nsubseteq \widehat{\mathfrak{m}}^2$ for all $n \geq 1$. Hence $(y^n, u^n, v^n)^{\lim} \nsubseteq \mathfrak{m}^2$ for all $n \geq 1$. This section is devoted to compute explicitly $(y^n, u^n, v^n)^{\lim}$, $n \geq 1$.

Discussion 8.2. Keep all notations as in the previous Example. We have $(y^n, u^n, v^n)^{\lim} \nsubseteq \mathfrak{m}^2$ for all $n \geq 1$. By the definition of the limit closure for each n there exists $t(n)$ such that

$$(y^{nt(n)+n}, u^{nt(n)+n}, v^{nt(n)+n}) :_R (yuv)^{nt(n)} \nsubseteq \mathfrak{m}^2.$$

For all $n = 1, \ldots, 9$, by using computer program we can compute that

$$(y^{2n}, u^{2n}, v^{2n}) :_R (yuv)^n = (y^n, u^n, v^n, a_n),$$

where a_n as in the following table.

n	a_n
1	x
2	$x - yv$
3	$x - yv - yuv^2$
4	$x - yv - yuv^2 - 2yu^2v^3$
5	$x - yv - yuv^2 - 2yu^2v^3 - 5yu^3v^4$
6	$x - yv - yuv^2 - 2yu^2v^3 - 5yu^3v^4 - 14u^4v^5$
7	$x - yv - yuv^2 - 2yu^2v^3 - 5yu^3v^4 - 14u^4v^5 - 42yu^5v^6$
8	$x - yv - yuv^2 - 2yu^2v^3 - 5yu^3v^4 - 14u^4v^5 - 42yu^5v^6 - 132yu^6v^7$
9	$x - yv - yuv^2 - 2yu^2v^3 - 5yu^3v^4 - 14u^4v^5 - 42yu^5v^6 - 132yu^6v^7 - 429yu^7v^8$

From the above table, we consider the sequence 1, 1, 2, 5, 14, 42, 132, 429, ... This is the first nine terms, from C_0 to C_8, of the Catalan sequence.

Definition 8.3. The *Catalan numbers*, C_i $(i \geq 0)$, are numbers satisfy the recurrence relation

$$C_0 = 1 \text{ and } C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \text{ for all } n \geq 0.$$

The following is well-known.

Lemma 8.4. *Let $C(t) = \sum_{i=0}^{\infty} C_i t^i$ be the generating function for Catalan numbers. Then*

$$C(t) = 1 + tC(t)^2.$$

Proposition 8.5. *With notations as in Example 8.1, and $C(t) = \sum_{i=0}^{\infty} C_i t^i$ is the generating function for Catalan numbers. Then*

$$(X - YVC(UV)).(Y - XUC(UV)) = (XY - UX^2 - VY^2).C(UV).$$

Therefore $(x - yvC(uv)) \cap (y - xuC(uv)) = (0)$ is a reduced primary decomposition of (0) in $\widehat{R}$.

Proof. We have

$$\begin{aligned}(X - YVC(UV)).(Y - XUC(UV)) &= XY - (UX^2 + VY^2).C(UV) + XYUV.C(UV)^2 \\ &= XY.(1 + UVC(UV)^2) - (UX^2 + VY^2).C(UV) \\ &= (XY - UX^2 - VY^2).C(UV) \quad \text{(Lemma 8.4)}.\end{aligned}$$

Hence $(x - yvC(uv)).(y - xuC(uv)) = 0 \in \widehat{R}$. It is easy to check that $(x - yvC(uv)), (y - xuC(uv))$ are prime ideals and $(x - yvC(uv)) \cap (y - xuC(uv)) = (0)$. ☐

The following is the main result of this section.

Proposition 8.6. *Let K be a field of characteristic 0 and let $A = K[X, Y, U, V]/(f)$ where $f = XY - UX^2 - VY^2$. We denote by x, y, u, v the images of X, Y, U, V, respectively. Set $R = A_{\mathfrak{m}}$ where $\mathfrak{m} = (x, y, u, v)$. Then for all $n \geq 1$ we have*

$$(y^n, u^n, v^n)^{\lim} = (y^n, u^n, v^n, x - yv\sum_{i=0}^{n-2} C_i(uv)^i),$$

where C_i is the i-th Catalan number.

Lemma 8.7. *Keep all notations as in the previous Proposition. Then for all $n \geq 1$ we have*

$$(y^n, u^n, v^n, x - yv\sum_{i=0}^{n-2} C_i(uv)^i) \subseteq (y^{2n}, u^{2n}, v^{2n}) :_R (yuv)^n.$$

Proof. It suffices to prove for all $n \geq 1$ that

$$(x - yv\sum_{i=0}^{n-2} C_i(uv)^i)y^n \in (u^n, v^n).$$

By Proposition 8.5 we have $(x - yvC(uv)).(y - xuC(uv)) = 0$. Hence

$$(x - yv\sum_{i=0}^{n-2} C_i(uv)^i).(y - xu\sum_{i=0}^{n-2} C_i(uv)^i) \equiv 0 \,(mod\,(u^n, v^n)).$$

Therefore

$$(x - yv\sum_{i=0}^{n-2} C_i(uv)^i).y \equiv (x - yv\sum_{i=0}^{n-2} C_i(uv)^i).(xu\sum_{i=0}^{n-2} C_i(uv)^i)\,(mod\,(u^n, v^n)).$$

So

$$\begin{aligned}(x - yv\sum_{i=0}^{n-2} C_i(uv)^i).y^n &\equiv (x - yv\sum_{i=0}^{n-2} C_i(uv)^i).(xu\sum_{i=0}^{n-2} C_i(uv)^i)^n\,(mod\,(u^n, v^n))\\ &\equiv 0\ (mod\,(u^n, v^n)).\end{aligned}$$

The Lemma is proved. $\square$

Proof of Proposition 8.6. By Lemma 8.7, for all $n \geq 1$, we have

$$(y^n, u^n, v^n, x - yv\sum_{i=0}^{n-2} C_i(uv)^i) \subseteq (y^n, u^n, v^n)^{\lim}.$$

By Proposition 8.5 we have

$$(X-YV\sum_{i=0}^{n-2} C_i(UV)^i).(Y-XU\sum_{i=0}^{n-2} C_i(UV)^i) \equiv (XY-UX^2-VY^2).C(UV)\,(mod\,(U^n, V^n)).$$

So $f \in (Y^n, U^n, V^n, X - YV\sum_{i=0}^{n-2} C_i(UV)^i)$. Hence

$$\begin{aligned}\ell(R/(y^n, u^n, v^n, x - yv\sum_{i=0}^{n-2} C_i(uv)^i)) &= \ell(K[X,Y,U,V])/(Y^n, U^n, V^n, X - YV\sum_{i=0}^{n-2} C_i(UV)^i))\\ &= n^3.\end{aligned}$$

On the other hand

$$\ell(R/(y^n, u^n, v^n)^{\lim}) = \ell(\widehat{R}/(y^n, u^n, v^n)^{\lim}_{\widehat{R}}).$$

By Corollary 3.5 we have $\cap_{n\geq 1}(y^n, u^n, v^n)^{\lim}_{\widehat{R}} = (x - yvC(uv))$. Set $R' = \widehat{R}/(x - yvC(uv))$. By Proposition 2.7 we have

$$\begin{aligned}\ell(\widehat{R}/(y^n, u^n, v^n)^{\lim}_{\widehat{R}}) &= \ell(R'/(y^n, u^n, v^n)^{\lim}_{R'})\\ &= \ell(K[[Y,U,V]]/(Y^n, U^n, V^n)) = n^3.\end{aligned}$$

Therefore

$$\ell(R/(y^n, u^n, v^n, x - yv\sum_{i=0}^{n-2} C_i(uv)^i)) = \ell(R/(y^n, u^n, v^n)^{\lim}).$$

Thus

$$(y^n, u^n, v^n)^{\lim} = (y^n, u^n, v^n, x - yv\sum_{i=0}^{n-2} C_i(uv)^i)$$

for all $n \geq 1$. The proof is complete. $\square$

The next example shows that the condition $(R, \mathfrak{m})$ is complete in Corollary 3.5 is necessary.

Example 8.8. Let k be a field of characteristic 0 and let $A = K[X, Y, U, V]/((f) \cap (X))$ where $f = XY - UX^2 - VY^2$. We denote by x, y, u, v the images of X, Y, U, V, respectively. Set $R = A_{\mathfrak{m}}$ where $\mathfrak{m} = (x, y, u, v)$. By Proposition 8.5

$$(x) \cap (x - yvC(uv)) \cap (y - xuC(uv)) = (0)$$

is a reduced primary decomposition of (0) in $\widehat{R}$. Corollary 3.5 implies that

$$\cap_{n \geq 1} (y^n, u^n, v^n)^{\lim}_{\widehat{R}} = (x) \cap (x - yvC(uv)).$$

So $\cap_{n \geq 1} (y^n, u^n, v^n)^{\lim} = (0)$. But

$$J = \{\mathfrak{p} \in \mathrm{Ass} R \,:\, y, u, v \text{ is a system of parameters of } R/\mathfrak{p}\} = \{(x)\}$$

and $\cap_{\mathfrak{p} \cap \in J} N(\mathfrak{p}) = (x) \neq (0)$.

References

[1] M. Brodmann and R.Y. Sharp, *Local cohomology: An algebraic introduction with geometric applications,* Cambridge University Press. **60** (1998).
[2] W. Brus and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, 1998.
[3] C. Chevalley, On the theory of local rings, *Ann. of Math.* **44** (1943), 690–708.
[4] N.T. Cuong, On the least degree of polynomials bounding above the differences between lengths and multiplicities of certain system of parameters in local rings, *Nagoya Math. J.* **125** (1992), 105–114.
[5] N.T. Cuong and D.T. Cuong, *dd*-Sequences and partial Euler-Poincar characteristics of Koszul complex, *J. Algebra Appl.* **6** (2007), 207–231.
[6] N.T. Cuong and D.T. Cuong, On sequentially Cohen-Macaulay modules, *Kodai Math. J.* **30** (2007), 409–428.
[7] N.T. Cuong and D.T. Cuong, Local cohomology annihilators and Macaulayfication, *Acta. Math. Vietnam*, to appear.
[8] N.T. Cuong, N.T. Hoa and N.T.H. Loan, On centain length function associate to a system of parameters in local rings, *Vietnam. J. Math.* **27** (1999), 259–272.
[9] N. T. Cuong and N. D. Minh, Lengths of generalized fractions of modules having small polynomial type, *Math. Proc. Camb. Phil. Soc.* **128** (2000), 269–282.
[10] N.T. Cuong, M. Morales and L.T. Nhan, On the length of generalized fractions, *J. Algebra* **265** (2003), 100–113.
[11] N.T. Cuong and L.T. Nhan, Pseudo Cohen-Macaulay and pseudo generalized Cohen-Macaulay module, *J. Algebra*, **267** (2003), 156–177.
[12] S. Dutta and P. Roberts, A characterization of systems of parameters, *Proc. Amer. Math. Soc.* **124** (1996), 671–675.
[13] L. Fouli and C. Huneke, What is a system of parameters, *Proc. Amer. Math. Soc.* **139** (2011), 2681–2696.
[14] A. Grothendieck, *Local Cohomology*, Lect. Notes Math. vol. 41, Springer-Verlag, Berlin, 1967.
[15] R. Hartshorne, A property of *A*-sequence, *Bull. Math. Soc. France* **4** (1966), 61–66.
[16] R.C. Heitmann, The direct summand conjecture in dimension three, *Ann. of Math.* **156** (2002), 695–712.
[17] M. Hochster, Contracted ideals from integral extensions of regular rings, *Nagoya Math. J.* **51** (1973), 25–43.
[18] M. Hochster, Homological conjectures, old and new, *Illinois J. Math.* **51** (2007), 151–169.
[19] M. Hochster and C. Huneke, Indecomposable canonical modules and connectedness, in Commutative Algebra: Syzygies, Multiplicities and Birational Algebra, *Contemp. Math.* **159** (1994), 197–208.
[20] C. Huneke, Tight closure, parameter ideals, and geometry, in: *Six Lectures on Commutative Algebra* J. Elias, J.M. Giral, R.M. Miró-Roig, S. Zarzuela (ed.), Progress in Mathematics, vol. 166, Birkhäuser Verlag, Basel, 1998, 187–239.

[21] C. Huneke, Lectures on local cohomology (with an appendix by Amelia Taylor), *Contemp. Math.* **436** (2007), 51–100.
[22] C. Huneke and I. Swanson, *Integral Closure of Ideals, Rings, and Modules*, London Math. Soc. Lecture Note Series **336**, Cambridge University Press, 2006.
[23] H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.
[24] M. Morales and P.H. Quy, A study of the length function of generalized fractions of modules, *Proc. Edinburgh Math. Soc.*, to appear.
[25] M. Nagata, *Local rings*, Interscience, New York, 1962.
[26] P.H. Quy and K. Shimomoto, *F-injectivity and Frobenius closure of ideals in Noetherian rings of characteristic* $p > 0$, preprint, ArXiv: 1601.02524[math.AC].
[27] L.J. Ratliff, A theorem on prime divisors of zero and characterizations of unmixed local domains, *Pacific J. Math.* **65** (1976), 449–470.
[28] L.J. Ratliff, Powers of ideals in locally unmixed Noetherian rings, *Pacific J. Math.* **107** (1983), 459–472.
[29] P. Schenzel, On the dimension filtration and Cohen-Macaulay filtered modules, in: *Proc. of the Ferrara Meeting in Honor of Mario Fiorentini*, University of Antwerp, Wilrijk, Belgium, 1998, 245–264.
[30] J.R. Strooker, *Homological question in local algebra*, Lodon Math. Soc. Lecture Note Series, **145**, Cambridge University Press, 1990.
[31] N.V. Trung, A characterization of two-dimensional unmixed local rings, *Math. Proc. Camb. Phil. Soc.* **89** (1981), 237–239.

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