# A study of the length function of generalized fractions of modules [1]

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**Abstract**

Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M$ a finitely generated $R$-module of dimension $d$. Let $\underline{x} = x_1, ..., x_d$ be a system of parameters of $M$ and $\underline{n} = (n_1, ..., n_d)$ a $d$-tuple of positive integers. In this paper we study the length of generalized fractions $M(1/(x_1, ..., x_d, 1))$ which was introduced by Sharp and Hamieh in [24]. First, we study the growth of the function $J_{\underline{x},M}(\underline{n}) = \ell(M(1/(x_1^{n_1}, ..., x_d^{n_d}, 1))) - n_1...n_d e(\underline{x}; M)$. Then we give an explicit calculation for the function $J_{\underline{x},M}(\underline{n})$ in the case where $M$ admits a Macaulayfication. Most previous results on this topic are now easy to understand and to improve.

## 1 Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring and $M$ a finitely generated $R$-module of dimension $d$. Let $\underline{x} = x_1, ..., x_d$ be a system of parameters of $M$. In this paper we study the length of generalized fractions $M(1/(x_1, ..., x_d, 1))$ which was introduced by Sharp and Hamieh in [24]. It has been proved in [8, Lemma 2.3] that $M/((\underline{x})_M^{\lim})$ is isomorphic to $M(1/(x_1, ..., x_d, 1))$, where

$$(\underline{x})_M^{\lim} = \bigcup_{n>0} \big((x_1^{n+1}, ..., x_d^{n+1})M : (x_1...x_d)^n\big).$$

We call $(\underline{x})_M^{\lim}$ the *limit closure*. If $M = R$ we write $(\underline{x})^{\lim}$.

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It should be noted that the Hochster monomial conjecture is equivalent to the claim $(\underline{x})^{\lim} \neq R$ for all system of parameters $\underline{x}$.

Let $\underline{n} = (n_1, ..., n_d)$ be a $d$-tuple of positive integers and $\underline{x}^{\underline{n}} = x_1^{n_1}, ..., x_d^{n_d}$. We consider the functions in $\underline{n}$,

$$I_{\underline{x},M}(\underline{n}) = \ell(M/(\underline{x}^{\underline{n}})M) - e(\underline{x}^{\underline{n}}; M),$$

$$J_{\underline{x},M}(\underline{n}) = e(\underline{x}^{\underline{n}}, M) - \ell(M/(\underline{x}^{\underline{n}})^{\lim}_M),$$

where $e(\underline{x}; M)$ is the Serre multiplicity of $M$ with respect to the sequence $\underline{x}$. In several papers N.T. Cuong et als, showed that the least degree of all polynomials in $\underline{n}$ bounding above $I_{\underline{x},M}(\underline{n})$ is independent of the choice of $\underline{x}$. It is called *the polynomial type* of $M$, and denoted by $p(M)$. The behavior of the function $J_{\underline{x},M}(\underline{n})$ was studied in [20] and [10]. In general $J_{\underline{x},M}(\underline{n})$ is not a polynomial in $\underline{n}$. Furthermore, the least degree of polynomials bounding above $J_{\underline{x},M}(\underline{n})$ is independent of the choice of $\underline{x}$. (see [7, Theorem 4.4]). It is called *the polynomial type of generalized fractions* of $M$, and denoted by $pf(M)$.

These two functions are closely related. In general it was proved in [21, Theorem 4.5] that $pf(M) \leq p(M)$. Our first result proves that if $M$ is unmixed and $\underline{x}$ is a certain system of parameters, then $I_{\underline{x},M}(\underline{n}) \leq 2^{d-2} J_{\underline{x},M}(\underline{n})$, which implies that $pf(M) = p(M)$.

Our second result consist to study the function $J_{\underline{x},M}(\underline{n})$ in the case where $M$ admits a Macaulayfication and we can express $J_{\underline{x},M}(\underline{n})$ in terms of the Non Cohen-Macaulay locus of $M$. As an application, in characteristic $p > 0$, we establish a connection between $J_{\underline{x},M}(\underline{n})$ and the Hilbert-Kunz function, and prove by using a recent result of Brenner [1], the existence of a local ring and a system of parameters such that the function $J_{\underline{x},M}(\underline{n})$, with $n = n_1 = ... = n_d$, cant be defined by a finite set of polynomials.

# 2 Preliminaries

First we recall the notion of *polynomial type* of a module. Let $(R, \mathfrak{m})$ be a Noetherian local ring, $M$ a finitely generated $R$-module of dimension $d$, $\underline{x} = x_1, ..., x_d$ a system of parameters of $M$, and $\underline{n} = (n_1, ..., n_d)$ a $d$-tuple of positive integers. We set $\underline{x}^{\underline{n}} = x_1^{n_1}, ..., x_d^{n_d}$ and we consider the function in $\underline{n}$

$$I_{\underline{x},M}(\underline{n}) = \ell(M/(\underline{x}^{\underline{n}})M) - e(\underline{x}^{\underline{n}}; M),$$

where $e(\underline{x}; M)$ is the Serre multiplicity of $M$ with respect to the sequence $\underline{x}$. N.T. Cuong in [3, Theorem 2.3] showed that the least degree of all polynomials in $\underline{n}$ bounding above $I_{\underline{x},M}(\underline{n})$ is independent of the choice of $\underline{x}$.

**Definition 2.1.** The least degree of all polynomials in $\underline{n}$ bounding above $I_{\underline{x},M}(\underline{n})$ is called *the polynomial type* of $M$, and is denoted by $p(M)$.

The following basic properties of $p(M)$ can be found in [3].

**Remark 2.2.** (i) We have $p(M) = p(\widehat{M}) \leq d-1$, where $\widehat{M}$ is the $\mathfrak{m}$-adic completion of $M$.

(ii) An $R$-module $M$ is Cohen-Macaulay if and only if $p(M) = -\infty$. Moreover, $M$ is generalized Cohen-Macaulay if and only if $p(M) \leq 0$.

Let $\mathfrak{a}_i(M) = \mathrm{Ann}H^i_{\mathfrak{m}}(M)$ for $0 \leq i \leq d-1$ and $\mathfrak{a}(M) = \mathfrak{a}_0(M) \cdots \mathfrak{a}_{d-1}(M)$. We denote by $NC(M)$ the non-Cohen-Macaulay locus of $M$ i.e. $NC(M) = \{\mathfrak{p} \in \mathrm{supp}(M) \,|\, M_{\mathfrak{p}}$ is not Cohen-Macaulay$\}$. Recall that $M$ is called *equidimensional* if $\dim M = \dim R/\mathfrak{p}$ for all minimal associated primes of $M$. The polynomial type of a module can be well understood by the annihilator of local cohomology as follows.

**Proposition 2.3** ([2], Theorem 1.2)**.** *Suppose that $R$ admits a dualizing complex. Then*

(i) $p(M) = \dim R/\mathfrak{a}(M)$.

(ii) *If $M$ is equidimensional then $p(M) = \dim(NC(M))$.*

Although the function $I_{\underline{x},M}(\underline{n})$ is not a polynomial in general, it has a good behavior for some special systems of parameters.

**Definition 2.4** ([4])**.** A system of parameters $x_1, ..., x_d$ of $M$ is called *p-standard* if $x_d \in \mathfrak{a}(M)$ and $x_i \in \mathfrak{a}(M/(x_{i+1}, ..., x_d)M)$ for all $i = d-1, ..., 1$.

**Definition 2.5** ([17], [16])**.** (i) A sequence in $R$, $\underline{x} = x_1, ..., x_s$ is called a *d-sequence* of $M$ if $(x_1, ..., x_{i-1})M : x_j = (x_1, ..., x_{i-1})M : x_i x_j$ for all $i \leq j \leq s$.

(ii) A sequence $\underline{x} = x_1, ..., x_s$ is called a *strong d-sequence* if $\underline{x}^{\underline{n}} = x_1^{n_1}, ..., x_s^{n_s}$ is a $d$-sequence for all $\underline{n} = (n_1, ..., n_s) \in \mathbb{N}^s$.

For important properties of $d$-sequence, see [17] and [26].

**Definition 2.6** ([5])**.** A sequence of elements $\underline{x} = x_1, ..., x_s$ is called a *dd-sequence* of $M$ if $\underline{x}$ is a strong $d$-sequence of $M$ and the following conditions are satisfied:

(i) $s = 1$ or,

(ii) $s > 1$ and $\underline{x}' = x_1, ..., x_{s-1}$ is a $dd$-sequence of $M/x_s^n$ for all $n \geq 1$.

The function $I_{\underline{x},M}(\underline{n})$ is a polynomial for a $p$-standard system of parameters or $dd$-sequence of parameters (see [4, Theorem 2.6 (ii)] and [5, Theorem 1.2]).

**Proposition 2.7.** *A system of parameters $\underline{x} = x_1, ..., x_d$ of $M$ is a dd-sequence iff for all $n_1, ..., n_d > 0$ we have*

$$I_{\underline{x},M}(\underline{n}) = \sum_{i=0}^{p(M)} n_1...n_i e_i,$$

*where $e_i = e(x_1, ..., x_i; 0 :_{M/(x_{i+2},...,x_d)M} x_{i+1})$ and $e_0 = \ell(0 :_{M/(x_2,...,x_d)M} x_1)$. Moreover a p-standard system of parameters is a dd-sequence system of parameters.*

In order to introduce the notion of *polynomial type of generalized fractions* we recall the notion of *limit closure* of a parameter ideal.

**Definition 2.8.** Let $\underline{x} = x_1, ..., x_d$ be a system of parameters of $M$. Then the *limit closure* of $\underline{x}$ in $M$ is a submodule of $M$ defined by

$$(\underline{x})_M^{\lim} = \bigcup_{n>0} \big((x_1^{n+1}, ..., x_d^{n+1})M : (x_1...x_d)^n\big),$$

when $M = R$ we write $(\underline{x})^{\lim}$ for short.

For a study of limit closure we refer to [13].

**Remark 2.9.** (i) It is well known that $(\underline{x})M = (\underline{x})_M^{\lim}$ if and only if $\underline{x}$ is an $M$-sequence i.e. $M$ is Cohen-Macaulay.

(ii) The quotient $(\underline{x})_M^{\lim}/(\underline{x})M$ is the kernel of the canonical map

$$H^d(\underline{x}; M) \to H^d_{\mathfrak{m}}(M).$$

(iii) (see [6, Lemma 2.4]) If $x_1, ..., x_d$ is a $dd$-sequence we have

$$(\underline{x})_M^{\lim} = \sum_{i=1}^{d} \big[(x_1, ..., \widehat{x_i}, ..., x_d)M :_M x_i\big] + (\underline{x})M.$$

Similarly to the notion of polynomial type, we consider the function in $\underline{n}$

$$J_{\underline{x},M}(\underline{n}) = e(\underline{x}^{\underline{n}}, M) - \ell(M/(\underline{x}^{\underline{n}})_M^{\lim}).$$

In general $J_{\underline{x},M}(\underline{n})$ is not a polynomial in $\underline{n}$ (cf. [10]) but it is bounded by polynomials. Furthermore, the least degree of polynomials bounding above $J_{\underline{x},M}(\underline{n})$ is independent of the choice of $\underline{x}$ (see [7, Theorem 4.4]).

**Definition 2.10.** The least degree of all polynomials in $\underline{n}$ bounding above $J_{\underline{x},M}(\underline{n})$ is called *the polynomial type of generalized fractions* of $M$, and denoted by $pf(M)$.

Now we recall the notion of *unmixed component* of $M$ which is closely related with the limit closure and the polynomial type of generalized fractions.

**Definition 2.11.** The largest submodule of $M$ of dimension less than $d$ is called *the unmixed component* of $M$ and it is denoted by $U_M(0)$.

It should be noted that if $\cap_{\mathfrak{p}\in \mathrm{Ass}M} N(\mathfrak{p}) = 0_M$ is a reduced primary decomposition of the zero submodule of $M$, then $U_M(0) = \cap_{\mathfrak{p}\in \mathrm{Assh}M} N(\mathfrak{p})$, where $\mathrm{Assh}M = \{\mathfrak{p} \in \mathrm{Ass}M \mid \dim R/\mathfrak{p} = \dim M\}$.

**Remark 2.12.** (i) In [13, Theorem 4.1] it is proved that $U_M(0) = \cap_n (\underline{x}^{[n]})_M^{\lim}$ for any system of parameters $\underline{x}$ of $M$, where we denote $\underline{x}^{[n]} = x_1^n, ..., x_d^n$.

(ii) (cf. [11, Theorem 3.1]) Suppose that $R$ admits a dualizing complex then $pf(M) = -\infty$ (resp. $pf(M) \leq 0$) if and only if $M/U_M(0)$ is Cohen-Macaulay (resp. generalized Cohen-Macaulay).

Recently, N.T. Cuong and the second author study the splitting of local cohomology (cf. [12], [14]), this will provide the main tool for the proof of our first result in this paper. We collect here some results which we need in the sequel. Set

$$\mathfrak{b}(M) = \bigcap_{\underline{x};i=1}^{d} \mathrm{Ann}(0 : x_i)_{M/(x_1,...,x_{i-1})M},$$

where $\underline{x} = x_1, ..., x_d$ runs over all systems of parameters of $M$. By [23, Satz 2.4.5] we have

$$\mathfrak{a}(M) \subseteq \mathfrak{b}(M) \subseteq \mathfrak{a}_0(M) \cap \cdots \cap \mathfrak{a}_{d-1}(M).$$

We have the following splitting property.

**Theorem 2.13** ([14], Corollary 3.5). *Let $x \in \mathfrak{b}(M)^3$ be a parameter element of $M$. Let $U_M(0)$ be the unmixed component of $M$ and set $\overline{M} = M/U_M(0)$. Then*

$$H^i_{\mathfrak{m}}(M/xM) \cong H^i_{\mathfrak{m}}(M) \oplus H^{i+1}_{\mathfrak{m}}(\overline{M})$$

*for all $i < d - 1$.*

**Lemma 2.14.** *Let $N \subseteq H^0_{\mathfrak{m}}(M)$ be a submodule of finite length. Then $\mathfrak{b}(M) \subseteq \mathfrak{b}(M/N)$.*

*Proof.* Let $x_1, ..., x_d$ be an arbitrary system of parameters of $M/N$. It is also a system of parameters of $M$. By definition of $\mathfrak{b}(M/N)$, we need only to prove that

$$\mathfrak{b}(M) \subseteq \mathrm{Ann}\frac{[(x_1, ..., x_{i-1})M + N] : x_i}{(x_1, ..., x_{i-1})M + N}$$

for all $i \leq d$. Choose a positive integer $n_0$ such that $x_i^{n_0}N = 0$ and for all $n \geq n_0$ we have

$$(x_1, ..., x_{i-1})M : x_i^n = (x_1, ..., x_{i-1})M : x_i^{n_0},$$
$$[(x_1, ..., x_{i-1})M + N] : x_i^n = [(x_1, ..., x_{i-1})M + N] : x_i^{n_0}.$$

So

$$[(x_1, ..., x_{i-1})M + N] : x_i^{n_0} \subseteq (x_1, ..., x_{i-1})M : x_i^{2n_0} \subseteq [(x_1, ..., x_{i-1})M + N] : x_i^{2n_0}.$$

Hence $(x_1, ..., x_{i-1})M : x_i^{2n_0} = [(x_1, ..., x_{i-1})M + N] : x_i^{2n_0}$ and we have

$$\begin{aligned} \mathrm{Ann}\frac{[(x_1, ..., x_{i-1})M + N] : x_i}{(x_1, ..., x_{i-1})M + N} &\supseteq \mathrm{Ann}\frac{[(x_1, ..., x_{i-1})M + N] : x_i^{2n_0}}{(x_1, ..., x_{i-1})M + N} \\ &= \mathrm{Ann}\frac{(x_1, ..., x_{i-1})M : x_i^{2n_0}}{(x_1, ..., x_{i-1})M + N} \\ &\supseteq \mathrm{Ann}\frac{(x_1, ..., x_{i-1})M : x_i^{2n_0}}{(x_1, ..., x_{i-1})M} \\ &\supseteq \mathfrak{b}(M). \end{aligned}$$

□

The following notion of system of parameters is closed related with $p$-standard and $dd$-sequence system of parameters and very useful in this paper.

**Definition 2.15.** A system of parameters $x_1, ..., x_d$ is called a $C$-system of parameters of $M$ if $x_d \in \mathfrak{b}(M)^3$ and $x_i \in \mathfrak{b}(M/(x_{i+1}, ..., x_d)M)^3$ for all $i = d-1, ..., 1$.

We call $C$-system of parameters in honor of Professor N.T. Cuong. If $(R, \mathfrak{m})$ is the quotient of a Cohen-Macaulay ring then we always have that $\dim R/\mathfrak{a}(M) < \dim M$ for every finitely generated $R$-module $M$. So every finitely generated $R$-module $M$ admits a $C$-system of parameters.

**Lemma 2.16.** *Let $x_1, ..., x_d$ be a $C$-system of parameters of $M$. Then*

(i) $x_1, ..., x_d$ is a $dd$-sequence.

(ii) $x_1^{n_1}, ..., x_d^{n_d}$ is a $C$-system of parameters of $M$ for all $n_1, ...., n_d \geq 1$.

(iii) For all $i \leq d$ we have $x_1, ..., x_{i-1}, x_{i+1}, ..., x_d$ is a $C$-system of parameters of $M/x_iM$.

(iv) Let $N \subseteq H^0_{\mathfrak{m}}(M)$ be a submodule of finite length. Then $x_1, ..., x_d$ is a $C$-system of parameters of $M/N$.

*Proof.* (i) is [14, Proposition 4.6], (ii) is [14, Corollary 4.5] and (iii) is [14, Lemma 2.10].
(iv) For each $i \leq d$ we have $M/((x_{i+1}, ..., x_d)M + N)$ is a quotient module of $M/(x_{i+1}, ..., x_d)M$ by a submodule of finite length. So $\mathfrak{b}(M/(x_{i+1}, ..., x_d)M) \subseteq \mathfrak{b}(M/((x_{i+1}, ..., x_d)M + N))$ by Lemma 2.14. Thus

$$x_i \in \mathfrak{b}(M/(x_{i+1}, ..., x_d)M)^3 \subseteq \mathfrak{b}(M/((x_{i+1}, ..., x_d)M + N))^3.$$

□

# 3 On the polynomial type of generalized fractions

Since $p(M)$ and $pf(M)$ do not change after passing to the completion. In this section we assume that $(R, \mathfrak{m})$ is the image of a Cohen-Macaulay local ring. For each system of parameters $\underline{x} = x_1, ..., x_d$ set

$$I_{\underline{x},M} = \ell(M/(\underline{x})M) - e(\underline{x}; M)$$

and

$$J_{\underline{x},M} = e(\underline{x}; M) - \ell(M/(\underline{x})^{\lim}_M).$$

It should be noted that $I_{\underline{x},M}$ is much easier to understand than $J_{\underline{x},M}$.

**Lemma 3.1.** *Let $M$ be a generalized Cohen-Macaulay module and $\underline{x} = x_1, ..., x_d$ a standard system of parameters of $M$. Then*

(i) $I_{\underline{x},M} = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(M))$.

(ii) $J_{\underline{x},M} = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell(H^i_{\mathfrak{m}}(M))$.

*Proof.* For the definition of standard system of parameters and the proof of (i) see [27], (ii) follows from [7, Theorem 5.1]. □

**Lemma 3.2.** *Let $\underline{x} = x_1, ..., x_d$ be a system of parameters of $M$ and $U_M(0)$ the unmixed component of $M$. Set $\overline{M} = M/U_M(0)$ we have*

(i) $J_{\underline{x},M} = J_{\underline{x},\overline{M}}$.

(ii) $J_{\underline{x},M}(\underline{n}) = J_{\underline{x},\overline{M}}(\underline{n})$ for all $\underline{n}$.

(iii) $pf(M) = pf(\overline{M})$.

*Proof.* (i) Since $\dim U_M(0) < d$ we have $e(\underline{x}; M) = e(\underline{x}; \overline{M})$. For each $n \geq 1$ we set $\underline{x}^{[n]} = x_1^n, ..., x_d^n$. By Remark 2.12 we have $U_M(0) = \cap_{n\geq 1}(\underline{x}^{[n]})_M^{\lim}$. By [13, Proposition 2.6] we have

$$\ell(M/(\underline{x})_M^{\lim}) = \ell(\overline{M}/(\underline{x})_{\overline{M}}^{\lim}).$$

Therefore $J_{\underline{x},M} = J_{\underline{x},\overline{M}}$.
(ii) follows from (i) and (iii) follows from (ii). □

By the above lemma, we can assume that $M$ is unmixed i.e. $U_M(0) = 0$, for the computation of either the function $J_{\underline{x},M}(\underline{n})$ or $pf(M)$. The following is important for our inductive technique.

**Remark 3.3.** Let $M$ be an unmixed finitely generated $R$-module of dimension $d$. Then

(i) $H^1_{\mathfrak{m}}(M)$ is finitely generated provided $d \geq 2$ (for example see [15, Lemma 3.1]).

(ii) The set

$$\mathcal{F}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \dim M_{\mathfrak{p}} > 1 = \operatorname{depth} M_{\mathfrak{p}},\ \mathfrak{p} \neq \mathfrak{m}\}$$

is finite (cf. [15, Lemma 3.2]).

(iii) Let $\underline{x} = x_1, ..., x_d$ be a $C$-system of parameters of $M$. Then

$$\mathcal{F}(M) = \operatorname{Ass} U_{M/x_dM}(0) \setminus \{\mathfrak{m}\}$$

and $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{F}(M)$. Hence $\operatorname{Ass} M/x_1M \subseteq \operatorname{Assh} M/x_1M \cup \{\mathfrak{m}\}$, so $U_{M/x_1M}(0) \cong H^0_{\mathfrak{m}}(M/x_1M)$ (cf. [14, Proposition 4.11, Remark 4.12]).

**Lemma 3.4.** *Let $M$ be an unmixed finitely generated $R$-module of dimension $d \geq 2$ and $\underline{x} = x_1, ..., x_d$ a $C$-system of parameters of $M$. Then $x_1.H^1_{\mathfrak{m}}(M) = 0$ and $\ell(H^1_{\mathfrak{m}}(M)) \leq I_{\underline{x},M}$.*

*Proof.* Set $M_d = M/x_dM$. Since $M$ is unmixed, by Theorem 2.13 we have $H^1_{\mathfrak{m}}(M) \cong H^0_{\mathfrak{m}}(M_d)$. By Lemma 2.16 we have $\underline{x}' = x_1, ..., x_{d-1}$ is a $dd$-sequence of $M_d$ so $H^0_{\mathfrak{m}}(M_d) = 0 :_{M_d} x_1$. Hence $x_1.H^1_{\mathfrak{m}}(M) = 0$. Moreover the properties of $dd$-sequences imply that $H^0_{\mathfrak{m}}(M_d) \cap (\underline{x}')M_d = 0$. Thus

$$\begin{aligned}\ell(M_d/(\underline{x}')M_d) = \ell(H^0_{\mathfrak{m}}(M_d)) + \ell(\overline{M_d}/(\underline{x}')\overline{M_d}) &\geq \ell(H^1_{\mathfrak{m}}(M)) + e(\underline{x}'; \overline{M_d}) \\ &= \ell(H^1_{\mathfrak{m}}(M)) + e(\underline{x}'; M_d),\end{aligned}$$

where $\overline{M_d} = M_d/H^0_{\mathfrak{m}}(M_d)$. Therefore

$$\ell(H^1_{\mathfrak{m}}(M)) \leq I_{\underline{x}',M_d} = I_{\underline{x},M}.$$

For the last equality notice that since $x_d$ is $M$-regular we have $e(\underline{x};M) = e(\underline{x}';M_d)$. The proof is complete. □

**Lemma 3.5.** *Let $M$ be an unmixed finitely generated $R$-module of dimension $d \geq 3$ and $\underline{x} = x_1, ..., x_d$ a $C$-system of parameters of $M$. Set $M_1 = M/x_1M$ and $\underline{x}' = x_2, ..., x_d$ we have $I_{\underline{x},M} \leq 2I_{\underline{x}',\overline{M}_1}$, where $\overline{M_1} = M_1/H^0_{\mathfrak{m}}(M_1)$.*

*Proof.* Since $x_1$ is $M$-regular we have $e(\underline{x};M) = e(\underline{x}';M_d)$. So $I_{\underline{x},M} = I_{\underline{x}',M_1}$. By Lemma 2.16 we have $\underline{x}' = x_2, ..., x_d$ is a $C$-system of parameters of $M_1$. Similar to the proof of the previous result we have

$$I_{\underline{x}',M_1} = I_{\underline{x}',\overline{M}_1} + \ell(H^0_{\mathfrak{m}}(M_1)).$$

Thus we need only to prove that $\ell(H^0_{\mathfrak{m}}(M_1)) \leq I_{\underline{x}',\overline{M}_1}$. Consider the following short exact sequence

$$0 \longrightarrow M \xrightarrow{x_1.} M \longrightarrow M_1 \longrightarrow 0.$$

By Lemma 3.4 we have $x_1.H^1_{\mathfrak{m}}(M) = 0$. So by applying the local cohomology functor to the above short exact sequence we have $H^0_{\mathfrak{m}}(M_1) \cong H^1_{\mathfrak{m}}(M)$ and

$$0 \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow H^1_{\mathfrak{m}}(M_1).$$

Thus

$$\ell(H^0_{\mathfrak{m}}(M_1)) = \ell(H^1_{\mathfrak{m}}(M)) \leq \ell(H^1_{\mathfrak{m}}(M_1)).$$

On the other hand by Remark 3.3 we have $\overline{M_1}$ is unmixed, and $\underline{x}'$ is a $C$-system of parameters of $\overline{M_1}$ by Lemma 2.16. So

$$\ell(H^1_{\mathfrak{m}}(M_1)) = \ell(H^1_{\mathfrak{m}}(\overline{M}_1)) \leq I_{\underline{x}',\overline{M}_1}$$

by Lemma 3.4. Thus $\ell(H^0_{\mathfrak{m}}(M_1)) \leq I_{\underline{x}',\overline{M}_1}$. The proof is complete. □

**Proposition 3.6.** *Let $M$ be an unmixed finitely generated $R$-module of dimension $d$ and $\underline{x} = x_1, ..., x_d$ a $C$-system of parameters of $M$. Then $I_{\underline{x},M} \leq 2^{d-2}J_{\underline{x},M}$.*

*Proof.* We proceed by induction on $d$. The case $d = 1$ is trivial since $M$ is Cohen-Macaulay. For $d = 2$ by Lemma 3.1 we have

$$I_{\underline{x},M} = \ell(H^1_{\mathfrak{m}}(M)) = J_{\underline{x},M}.$$

Assume that $d \geq 3$ and the assertion was proved for $d - 1$. Set $M_1 = M/x_1M$ and $\underline{x}' = x_2, ..., x_d$ we have

$$\begin{aligned} I_{\underline{x},M} &\leq 2I_{\underline{x}',\overline{M}_1} && \text{(By Lemma 3.5)} \\ &\leq 2^{d-2}J_{\underline{x}',\overline{M}_1} && \text{(By induction)} \\ &= 2^{d-2}J_{\underline{x}',M_1} && \text{(By Lemma 3.2).} \end{aligned}$$

Since $x_1$ is $M$-regular we have $e(\underline{x}; M) = e(\underline{x}'; M_1)$. On the other hand we have

$$(\underline{x}')^{\lim}_{M_1} = \bigcup_n [(x_1, x_2^{n+1}, ..., x_d^{n+1})M :_M (x_2, ..., x_d)^n]/x_1M \subseteq (\underline{x})^{\lim}_M / x_1 M.$$

So $\ell(M/(\underline{x})^{\lim}_M) \leq \ell(M_1/(\underline{x}')^{\lim}_{M_1})$. Thus $J_{\underline{x}',M_1} \leq J_{\underline{x},M}$. Therefore we get the assertion $I_{\underline{x},M} \leq 2^{d-2} J_{\underline{x},M}$. □

**Theorem 3.7.** *Let $(R, \mathfrak{m})$ be the image of a Cohen-Macaulay local ring and $M$ an unmixed finitely generated $R$-module of dimension $d$. Then $pf(M) = p(M)$. Moreover $pf(M) = \dim R/\mathfrak{a}(M)$.*

*Proof.* By [21, Theorem 4.5] we have $pf(M) \leq p(M)$. Thus we need only to prove $pf(M) \geq p(M)$. Let $\underline{x} = x_1, ..., x_d$ be a $C$-system of parameters of $M$. By Lemma 2.16, for all $d$-tuples of positive integers $\underline{n} = (n_1, ..., n_d)$ we have $\underline{x}^{\underline{n}} = x_1^{n_1}, ..., x_d^{n_d}$ is also a $C$-system of parameters. By Proposition 3.6 we have

$$I_{\underline{x},M}(\underline{n}) = I_{\underline{x}^{\underline{n}},M} \leq 2^{d-2} J_{\underline{x}^{\underline{n}},M} = 2^{d-2} J_{\underline{x},M}(\underline{n})$$

for all $\underline{n} = (n_1, ..., n_d) \in \mathbb{N}^d$. Thus $p(M) \leq pf(M)$. The last assertion follows from Proposition 2.3. The proof is complete. □

The next result is a consequence of the above Theorem and Lemma 3.2.

**Corollary 3.8.** *Let $(R, \mathfrak{m})$ be the image of a Cohen-Macaulay local ring and $M$ a finitely generated $R$-module with the unmixed component $U_M(0)$. Then*

$$pf(M) = p(M/U_M(0)).$$

Recall that an $R$-module $M$ is called *pseudo (generalized) Cohen-Macaulay* if $pf(M) = 0$ (resp. $pf(M) \leq 0$). As a consequence of Corollary 3.8 we get a generalization of the main result of [11].

**Corollary 3.9.** *Let $(R, \mathfrak{m})$ be the image of a Cohen-Macaulay local ring and $M$ a finitely generated $R$-module with the unmixed component $U_M(0)$. Then $M$ is pseudo Cohen-Macaulay (resp. pseudo generalized Cohen-Macaulay) iff $M/U_M(0)$ if Cohen-Macaulay (resp. generalized Cohen-Macaulay).*

It is natural to raise the following question.

**Question 3.10.** Let $M$ be an unmixed finitely generated $R$-module of dimension $d$ and $\underline{x} = x_1, ..., x_d$ a $C$-system of parameters of $M$. Is it true that that the function $J_{\underline{x},M}(\underline{n})$ is a polynomial in $\underline{n}$ when $n_1, ..., n_d \gg 0$?

It should be noted that [9, Theorem 4.5] gives an affirmative answer for this question in the case $pf(M) \leq 1$.

## 4 The case $M$ admits a Macaulayfication

**Definition 4.1.** Let $M$ be a finitely generated $R$-module of dimension $d$. We say that $M$ admits a *Macaulayfication* $M'$ if we have an exact sequence

$$0 \to M \to M' \to N \to 0,$$

where $M'$ is a finitely generated Cohen-Macaulay $R$-module and $\dim N \leq d-2$.

**Remark 4.2** (see for example [19], [23])**.** Let $(R, \mathfrak{m})$ be a Noetherian complete local ring and $M$ a finitely generated $R$-module of dimension $d$. We recall that if $M$ is unmixed, the module $D^d(D^d(M))$ (where $D^d(M)$ is the Matlis dual of $M$) satisfies the condition $S_2$ and we have an exact sequence :

$$0 \to M \to D^d(D^d(M)) \to N \to 0$$

with $\dim N \leq d-2$. Moreover if there exist a finitely generated $R$-module $M'$ of dimension $d$, satisfying the condition $S_2$ and an exact sequence :

$$0 \to M \to M' \to M'/M \to 0$$

with $\dim M'/M \leq d-2$, then $M' \cong D^d(D^d(M))$. That is, if $M$ is unmixed the Macaulayfication is unique up to isomorphism (if exist). In this is the case, $\mathrm{Supp}(M'/M)$ is the non Cohen-Macaulay locus of $M$.

We can state the main result of this section.

**Theorem 4.3.** *Let $M$ be finitely generated $R$-module of dimension $d$. Suppose that $M$ has a Macaulayfication $M'$. Let $\underline{x} = x_1, ..., x_d$ be an arbitrary system of parameters of $M$. Set $N = M'/M$, then*

$$J_{\underline{x},M}(\underline{n}) = \ell(N/(\underline{x}^{\underline{n}})N)$$

*for all $d$-tuples $\underline{n} = (n_1, ..., n_d)$.*

*Proof.* For any system of parameters $\underline{y} = y_1, ..., y_d$, the short exact sequence

$$0 \to M \to M' \to N \to 0$$

induces the following commutative diagram with the last two columns exact

$$
\begin{array}{ccccccc}
 & & & & H^{d-1}(\underline{y};N) & \longrightarrow & H^{d-1}_{\mathfrak{m}}(N)=0 \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & (\underline{y})^{\lim}_M/(\underline{y})M & \xrightarrow{\beta} & H^d(\underline{y};M) & \longrightarrow & H^d_{\mathfrak{m}}(M) \\
 & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow \\
 & & 0 & \longrightarrow & H^d(\underline{y};M') & \longrightarrow & H^d_{\mathfrak{m}}(M') \\
 & & & & \downarrow & & \downarrow \\
 & & & & H^d(\underline{y};N) & \longrightarrow & H^d_{\mathfrak{m}}(N)=0 \\
 & & & & \downarrow & & \\
 & & & & 0. & &
\end{array}
$$

Both the second and the third rows are exact by Remark 2.9. Therefore we have $\alpha \circ \beta = 0$. Thus we have the following commutative diagram

$$
\begin{array}{ccccc}
0 & \longrightarrow & M/(\underline{y})^{\lim}_M & \xrightarrow{\pi} & H^d_{\mathfrak{m}}(M) \\
 & & \downarrow{\scriptstyle\overline{\alpha}} & & \downarrow{\scriptstyle\sigma} \\
0 & \longrightarrow & M'/(\underline{y})M' & \xrightarrow{\tau} & H^d_{\mathfrak{m}}(M') \\
 & & \downarrow & & \\
 & & N/(\underline{y})N) & & \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}
$$

with the middle column is exact. Moreover we have both $\pi$ and $\tau$ are injective and $\sigma$ is bijective. Therefore $\tau \circ \overline{\alpha} = \sigma \circ \pi$ is injective and so is $\overline{\alpha}$. Hence we have the following short exact sequence

$$0 \to M/(\underline{y})^{\lim}_M \to M'/(\underline{y})M' \to N/(\underline{y})N \to 0.$$

Thus

$$\ell(M/(\underline{y})^{\lim}_M) = \ell(M'/(\underline{y})M') - \ell(N/(\underline{y})N).$$

Now for each $\underline{n} = (n_1, ..., n_d)$, applying the above assertion for the system of parameters $\underline{x}^{\underline{n}} = x_1^{n_1}, ..., x_d^{n_d}$ we have

$$\ell(M/(\underline{x}^{\underline{n}})^{\lim}_M) = \ell(M'/(\underline{x}^{\underline{n}})M') - \ell(N/(\underline{x}^{\underline{n}})N).$$

Since $M'$ is Cohen-Macaulay we have

$$\ell(M'/(\underline{x}^{\underline{n}})M') = e(\underline{x}^{\underline{n}}; M') = e(\underline{x}^{\underline{n}}; M).$$

Therefore $J_{\underline{x},M}(\underline{n}) = \ell(N/(\underline{x}^{\underline{n}})N)$ for all $d$-tuples $\underline{n} = (n_1, ..., n_d)$. The proof is complete. $\square$

The length $\ell(N/(\underline{x}^{\underline{n}})N)$ is much easier to understand than the function $J_{\underline{x},M}(\underline{n})$. In many cases we can see that it coincides with a polynomial or a finite number of polynomials for $\underline{n} \gg 0$. The following Corollary extends [10, Lemma 2.4].

**Corollary 4.4.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 3$, $x_1, ..., x_d$ a system of parameters of $R$. Let $M = (x_1, ..., x_{d-v})$, $v \leq d-2$. Then for the system of parameters $\underline{x} = x_1 + x_d, x_2, ..., x_d$ of $M$ we have*

$$J_{\underline{x},M}(\underline{n}) = \ell(R/(x_1, ..., x_d))\ n_{d-v+1}...n_{d-1} \min\{n_1, n_d\}$$

*for all $n_1, ..., n_d \geq 1$. Therefore $J_{\underline{x},M}(\underline{n})$ is not a polynomial.*

*Proof.* Since $\dim R/M \leq d-2$, $R$ is a Macaulayfication of $M$. By Theorem 4.3 we have

$$J_{\underline{x},M}(\underline{n}) = \ell(R/(x_1, ..., x_{d-v}, (x_1 + x_d)^{n_1}, x_2^{n_2}, ..., x_d^{n_d})$$

for all $n_1, ..., n_d \geq 1$. Hence

$$\begin{aligned} J_{\underline{x},M}(\underline{n}) &= \ell(R/(x_1, ..., x_{d-v}, x_{d-v+1}^{n_{d-v+1}}, ..., x_{d-1}^{n_{d-1}}, x_d^{\min\{n_1, n_d\}}) \\ &= \ell(R/(x_1, ..., x_d))\ n_{d-v+1}...n_{d-1} \min\{n_1, n_d\} \end{aligned}$$

for all $n_1, ..., n_d \geq 1$. $\square$

The next result follows from Theorem 4.3 and Proposition 2.7.

**Corollary 4.5.** *Let $M$ be a finitely generated $R$-module of dimension $d$. Suppose that $M$ has a Macaulayfication $M'$ with $\dim M'/M = t$. Let $\underline{x} = x_1, ..., x_d$ be any system of parameters of $M$ such that $x_1, ..., x_t$ forms a dd-sequence of $N = M'/M$ and $x_{t+1}, ..., x_d \in \mathrm{Ann}N$. Then $J_{\underline{x},M}(\underline{n})$ is a polynomial in $\underline{n}$ for all $n_1, ..., n_d \geq 1$. Moreover*

$$J_{\underline{x},M}(\underline{n}) = n_1...n_t e(x_1, ..., x_t; N) + \sum_{i=0}^{t-1} n_1...n_i e_i,$$

*where $e_i = e(x_1, ..., x_i; 0 :_{N/(x_{i+2}, ..., x_t)N} x_{i+1})$ and $e_0 = \ell(0 :_{N/(x_2, ..., x_t)N} x_1)$.*

# 5 Relation with the Hilbert-Kunz function

By considering all explicit examples, it can be expected that $J_{\underline{x},M}(\underline{n})$ coincides with finitely many polynomials in $\underline{n}$ (cf. [10], [20]). As we will see this is not always the case. More precisely, we will give an example in characteristic $p$ such that the function $J_{\underline{x},M}(\underline{n})$ can not be controlled by finitely many polynomials. This question is closely related to the Hilbert-Kunz function.
Let $(A,\mathfrak{n})$ be a Noetherian local ring containing a field of positive characteristic $p$. Let $I$ be an ideal of $A$ and a prime power $q=p^e$ we define $I^{[q]}=(f^q|q\in I)$ as the $e$-th Frobenius power of $I$. If $I$ is an $\mathfrak{n}$-primary ideal we always have that $A/I^{[q]}$ has finite length. So we have a function

$$f_{HK}(I): q\mapsto \ell(A/I^{[q]}),$$

called the Hilbert-Kunz function, which was first studied by E. Kunz in [18]. In [22], P. Monsky proved that the limit

$$e_{HK}(I)=\lim_{q\to\infty}\frac{\ell(A/I^{[q]})}{q^{\dim A}}$$

exists as a real number; it is called the Hilbert-Kunz multiplicity of $I$, and the Hilbert-Kunz multiplicity of $\mathfrak{n}$ is also called the Hilbert-Kunz multiplicity of $A$. It is natural to ask whether the Hilbert-Kunz multiplicity of an $\mathfrak{n}$-primary ideal is always a rational number. There are many positive partial answers to this question. However, recently H. Brenner disproved this question by the following celebrate result.

**Theorem 5.1** ([1], Theorem 8.3)**.** *There exists a Noetherian local domain whose Hilbert-Kunz multiplicity is an irrational number.*

We are ready to prove the main result of this section.

**Theorem 5.2.** *There exist a regular local ring $(R,\mathfrak{m})$ of dimension $d$ with $\mathfrak{m}$ generates by a regular system of parameters $\underline{x}=(x_1,...,x_d)$ and a finitely generated $R$-module $M$, $\dim M=d$ such that the function $J_{\underline{x},M}(n)=n^d e(\underline{x};M)-\ell(M/(\underline{x}^{[n]})^{\lim}_M)$ can not be represented by finitely many polynomials in $n$, where $\underline{x}^{[n]}=x_1^n,...,x_d^n$.*

*Proof.* Let $(A,\mathfrak{n})$ be the ring of characteristic $p$ whose Hilbert-Kunz multiplicity is irrational as Brenner's result. Replacing $A$ by its completion, notice that the Hilbert-Kunz multiplicity does not change, we can assume that $(A,\mathfrak{n})$ is complete. By the Cohen structure theorem we have that $A$ is the image of a regular local ring $(R,\mathfrak{m})$ of dimension $d$. Since $e_{HK}(A)$ is irrational we have $A$ is not regular and so $\dim R-\dim A\geq 1$. If $\dim R-\dim A=1$ we replace $R$ by $R[X]_{(\mathfrak{m},X)R[X]}$. Henceforth we can assume that $\dim R-\dim A\geq 2$. Let the $R$-module $M$ be the kernel of the canonical map $R \to A$, we have $\dim M = d$. Choose a regular system of parameters $\underline{x} = x_1, ..., x_d$ generates $\mathfrak{m}$. By Theorem 4.3 we have

$$J_{\underline{x},M}(n) = \ell(A/(\underline{x}^{[n]})A)$$

for all $n \geq 1$. For all $i = 1, ..., d$ we denote by $a_i$ the image of $x_i$ in $A$. We have the sequence $\underline{a} = a_1, ..., a_d$ generates the maximal ideal $\mathfrak{n}$ of $A$. Now we assume that there are only finitely many polynomials $P_1(n), ..., P_r(n)$ such that for each $n \geq 1$ we have $J_{\underline{x},M}(n) = P_i(n)$ for some $i$ and find a contradiction. We consider the case $n$ is a prime power $q = p^e$ we have

$$J_{\underline{x},M}(q) = \ell(A/\underline{a}^{[q]}) = \ell(A/\mathfrak{n}^{[q]}).$$

Since there are infinitely many $q$, we must have a polynomial, says $P_1(n)$, such that

$$\ell(A/\mathfrak{n}^{[q]}) = P_1(q)$$

for infinitely many $q = p^e$. It should be noted that if a polynomial takes integer values at infinitely many integer numbers, then all of its coefficients are rational. Thus the leading coefficient of $P_1(n)$ is a rational number and $\deg P_1(n) = \dim A$. So

$$e_{HK}(A) = \lim_{q\to\infty} \frac{\ell(A/\mathfrak{n}^{[q]})}{q^{\dim A}} = \lim_{q\to\infty} \frac{P_1(q)}{q^{\dim A}}$$

is a rational number. It is a contradiction with our assumption about $A$. The proof is complete.

$\square$

For the next result we need the concept of the principle of *idealization*. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M$ a finitely generated $R$-module. We make the Cartesian product $R \times M$ into a commutative ring with respect to componentwise addition and multiplication defined by $(r, m) \cdot (r', m') = (rr', rm' + r'm)$. We call this the idealization of $M$ (over $R$) and denote it by $R \ltimes M$. The idealization $R \ltimes M$ is Noetherian local ring with identity $(1, 0)$, its maximal ideal is $\mathfrak{m} \times M$ and its Krull dimension is $\dim R$. If $\underline{x} = x_1, ..., x_d$ is a system of parameters of $R$ then $\underline{(x, 0)} = (x_1, 0), ..., (x_d, 0)$ is a system of parameters of the idealization $R \ltimes M$.

**Lemma 5.3** ([10], Lemma 2.6)**.** *Let $\dim M = \dim R = d$ and $S = R \ltimes M$. Let $\underline{x} = x_1, ..., x_d$ is a system of parameters of $R$. Then we have*

$$\ell(S/(\underline{x, 0})_S^{\lim}) = \ell(R/(\underline{x})_R^{\lim}) + \ell(M/(\underline{x})_M^{\lim}).$$

Now we prove the last result of this paper.

**Corollary 5.4.** *There exists a Noetherian local ring* $(S, \mathfrak{n})$ *of dimension* $d$ *and a system of parameters* $\underline{y} = y_1, ..., y_d$ *such that the function* $J_{\underline{y},S}(n)$ *can not be represented by finitely many polynomials in* $n$*.*

*Proof.* We choose $(R, \mathfrak{m})$ and $M$ as in Theorem 5.2. Let $\underline{x} = x_1, ..., x_d$ be a regular system of parameters of $R$. Let $S = R \ltimes M$ and $\underline{y} = (x_1, 0), ..., (x_d, 0)$. We can check that $e(\underline{y}; S) = e(\underline{x}; R) + e(\underline{x}; M)$. Since $R$ is regular we have $(\underline{x}^{[n]})_R^{\lim} = (\underline{x}^{[n]})$ for all $n$. So

$$\ell(R/(\underline{x}^{[n]})_R^{\lim}) = \ell(R/(\underline{x}^{[n]}) = n^d e(\underline{x}; R).$$

Combining with Lemma 5.3 we have

$$\begin{aligned}
J_{\underline{y},S}(n) &= \ell(S/(\underline{y}^{[n]})_S^{\lim}) - n^d e(\underline{y}; S) \\
&= \big(\ell(R/(\underline{x}^{[n]})_R^{\lim}) + \ell(M/(\underline{x}^{[n]})_M^{\lim})\big) - n^d\big(e(\underline{x}; R) + e(\underline{x}; M)\big) \\
&= \ell(M/(\underline{x}^{[n]})_M^{\lim}) - n^d e(\underline{x}; M) \\
&= J_{\underline{x},M}(n).
\end{aligned}$$

The assertion now follows from Theorem 5.2. The proof is complete. □

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