On the uniform bound of the index of reducibility of parameter ideals of a module whose polynomial type is at most one [1]

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Abstract

Let $(R, \mathfrak{m})$ be a Noetherian local ring, M a finitely generated R-module. The aim of this paper is to prove a uniform formula for the index of reducibility of paprameter ideals of M provided the polynomial type of M is at most one.

1 Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Notherian local ring, M a finitely generated R-module of dimension d. Let $\underline{x} = x_1, ..., x_d$ be a system of parameters of M and $\mathfrak{q} = (x_1, ..., x_d)$. Let $\underline{n} = (n_1, ..., n_d)$ be a d-tuple of positive integers and $\underline{x}^{\underline{n}} = x_1^{n_1}, ..., x_d^{n_d}$. We consider the difference

$$I_{M,\underline{x}}(\underline{n}) = \ell(M/(\underline{x}^{\underline{n}})M) - e(\underline{x}^{\underline{n}}; M)$$

as function in $\underline{n}$, where $e(\underline{x}; M)$ is the Serre multiplicity of M with respect to the sequence $\underline{x}$. Although $I_{M,\underline{x}}(\underline{n})$ may be not a polynomial for $n_1, ..., n_d$ large enough, it is bounded above by polynomials. Moreover, N.T. Cuong in [5] proved that the least degree of all polynomials in $\underline{n}$ bounding above $I_{M,\underline{x}}(\underline{n})$ is independent of the choice of $\underline{x}$, and it is denoted by $p(M)$. The invariant $p(M)$ is called *the polynomial type* of M. Recalling that M is a Cohen-Macaulay module if and only if $\ell(M/\mathfrak{q}M) = e(\mathfrak{q}; M)$ for some (and hence for all) parameter ideal $\mathfrak{q}$ of M. Thus, if we stipulate the degree of the zero polynomial is $-\infty$, then M is a Cohen-Macaulay module if and only if $p(M) = -\infty$. In order to generalize the class of Cohen-Macaulay module, J. Stuckrad and W. Vogel introduced the class of Buchsbaum modules. An R-module M is called Buchsbaum if and only if the difference $\ell(M/\mathfrak{q}M) - e(\mathfrak{q}; M)$ is a constant for all $\mathfrak{q}$. For the theory of Buchsbaum modules see [16]. Furthermore, N.T. Cuong, P. Schenzel and N.V. Trung introduced the class of generalized Cohen-Macaulay modules. Module M is generalized Cohen-Macaulay module if and only if the difference $\ell(M/\mathfrak{q}M) - e(\mathfrak{q}; M)$ is bounded above for all parameter ideals $\mathfrak{q}$. In that paper they showed that M is generalized Cohen-Macaulay if and only if the i-th local cohomology module $H^i_{\mathfrak{m}}(M)$ has finite length for all $i = 0, ..., d-1$. Set $I(M) = \sup_{\mathfrak{q}}\{\ell(M/\mathfrak{q}M) - e(\mathfrak{q}; M)\}$ where the supremum is taken over all parameter ideals of M. If M is a generalized Cohen-Macaulay module we have $I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(M))$, and this invariant is called the Buchsbaum invariant of M (see [8], [17]). It is easy to see that M is a generalized Cohen-Macaulay module if and only if $p(M) \leq 0$. The structure of M when $p(M) > 0$ is known little and there is no standard techniques to study since the local cohomology $H^i_{\mathfrak{m}}(M)$ may be not finitely generated for all $i \geq 1$. Even though the case $p(M) = 1$, the proof sometimes is very complicate (for example, see [1]).

Let $\mathfrak{q}$ be a parameter ideal of M. The number of irreducibility components appear in an irredundant irreducible decomposition of $\mathfrak{q}M$ is called the *index of reducibility* of $\mathfrak{q}$ on M, and denoted by $\mathcal{N}_R(\mathfrak{q}, M)$. It is well known that $\mathcal{N}_R(\mathfrak{q}, M) = \dim_{R/\mathfrak{m}} \mathrm{Soc}(M/\mathfrak{q}M)$, where $\mathrm{Soc}(N) = 0 :_N \mathfrak{m} \cong \mathrm{Hom}(R/\mathfrak{m}, N)$ for an arbitrary R-module N. A classical result of D.G. Northcott claimed that the index of reducibility of parameter ideals on a Cohen-Macaulay module is an invariant of

[1] **Key words and phrases:** The index of reducibility; the polynomial type of module; local cohomology
AMS Classification 2010: 13H10; 13D45.
This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2011.49.

the module. The converse of this result is not true, the first counterexample is given by S. Endo and M. Narita in [10]. If M is generalized Cohen-Macaulay, S. Goto and N. Suzuki proved that $\mathcal{N}_R(\mathfrak{q}, M)$ has an upper bound, more precisely

$$\mathcal{N}_R(\mathfrak{q}, M) \leq \sum_{i=0}^{d-1} \binom{d}{i} \ell(H^i_{\mathfrak{m}}(M)) + \dim_{R/\mathfrak{m}} \operatorname{Soc}(H^d_{\mathfrak{m}}(M)) \quad (\star)$$

for all paprameter ideals $\mathfrak{q}$ of M (cf. [12, Theorem 2.1]). It is worthy to mention that if M is Buchsbaum, Goto and H. Sakurai in [11] showed that the inequality $(\star)$ becomes an equality for all parameter ideals $\mathfrak{q}$ contained in a large enough power of $\mathfrak{m}$. In [9, Theorem 1.1], N.T. Cuong and H.L. Truong considered Goto-Sakurai's result for generalized Cohen-Macaulay modules. In fact, they proved that

$$\mathcal{N}_R(\mathfrak{q}, M) = \sum_{i=0}^{d} \binom{d}{i} \dim_{R/\mathfrak{m}} \operatorname{Soc}(H^i_{\mathfrak{m}}(M))$$

for all $\mathfrak{q} \subseteq \mathfrak{m}^n$, $n \gg 0$. Recently, N.T. Cuong and the author reproved this result based on the study of the splitting of local cohomology (cf. [7]). A generalization of Cuong-Truong's result can be found in [14].

The aim of this paper is to extend the result of Goto and Suzuki for the class of modules of the polynomial type at most one. We show that if M is a finitely generated R-module such that $p(M) \leq 1$, then $\mathcal{N}_R(\mathfrak{q}, M)$ is bounded above for all parameter ideals $\mathfrak{q}$ of M.

This paper is organized as follows. In Section 2 we recall the notions of the polynomial type of a module and the index of reducibility. This paper is inspired by the uniform property of the minimal number of generators of ideals in local rings of dimension one (cf. [15, Chapter 3]) which we mention in Section 3. By Matlis' dual we obtain a similar result for Artinian modules of dimension one. Based on this result we can give the proof of the main result by using the standard techniques of local cohomology in Section 4.

2 Preliminaries

We first recall the notion of *the polynomial type* of a module. Let $(R, \mathfrak{m})$ be a Notherian local ring, M a finitely generated R-module of dimension d. Let $\underline{x} = x_1, ..., x_d$ be a system of parameters of M and $\mathfrak{q} = (x_1, ..., x_d)$. Let $\underline{n} = (n_1, ..., n_d)$ be a d-tuple of positive integers and $\underline{x}^{\underline{n}} = x_1^{n_1}, ..., x_d^{n_d}$. We consider the difference

$$I_{M,\underline{x}}(\underline{n}) = \ell(M/(\underline{x}^{\underline{n}})M) - e(\underline{x}^{\underline{n}}; M)$$

as function in $\underline{n}$, where $e(\underline{x}; M)$ is the Serre multiplicity of M with respect to the sequence $\underline{x}$. N.T. Cuong in [5, Theorem 2.3] showed that the least degree of all polynomials in $\underline{n}$ bounding above $I_{M,\underline{x}}(\underline{n})$ is independent of the choice of $\underline{x}$.

Definition 2.1. The least degree of all polynomials in $\underline{n}$ bounding above $I_{M,\underline{x}}(\underline{n})$ is called *the polynomial type* of M, and denoted by $p(M)$.

The following basic properties of $p(M)$ can be found in [5].

Remark 2.2. (i) We have $p(M) \leq d - 1$.

(ii) An R-module M is Cohen-Macaulay if and only if $p(M) = -\infty$. Moreover, M is generalized Cohen-Macaulay if and only if $p(M) = 0$.

(iii) If we denote the $\mathfrak{m}$-adic completion of M by $\widehat{M}$, then $p(M) = p_{\widehat{R}}(\widehat{M})$.

Let $a_i(M) = \mathrm{Ann}H^i_{\mathfrak{m}}(M)$ for $0 \le i \le d-1$ and $\mathfrak{a}(M) = \mathfrak{a}_0(M)\cdots\mathfrak{a}_{d-1}(M)$. We denote by $NC(M)$ the non-Cohen-Macaulay locus of M i.e. $NC(M) = \{\mathfrak{p} \in \mathrm{supp}(M) \mid M_{\mathfrak{p}}$ is not Cohen-Macaulay$\}$. Recalling that M is called *equidimensional* if $\dim M = \dim R/\mathfrak{p}$ for all minimal associated primes of M. The following result give the meaning of the polynomial type.

Theorem 2.3 ([4], Theorem 1.2)**.** *Suppose that R admits a dualizing complex. Then*

(i) $p(M) = \dim R/\mathfrak{a}(M)$.

(ii) *If M is equidimensional then $p(M) = \dim(NC(M))$.*

Example 2.4. Let $S = k[[X_1, X_2, ..., X_{2n+1}]]/(X_1, ..., X_n) \cap (X_{n+1}, ..., X_{2n})$ where k is a field and n is a positive integer greater then 1. It is easy to see that R is not generalized Cohen-Macaulay but $p(M) = 1$.

Remark 2.5. By [13], there exists a local domain $(R, \mathfrak{m})$ of dimension two such that the $\mathfrak{m}$-adic completion of R is $\widehat{R} = k[[X, Y, Z]]/(X) \cap (Y, Z)$. We can check that $\dim R/\mathfrak{a}(R) = 2$ and $\dim(NC(R)) = 0$ but $\dim \widehat{R}/\mathfrak{a}(\widehat{R}) = \dim(NC(\widehat{R})) = p(\widehat{R}) = 1$. So $\dim R/\mathfrak{a}(R)$ and $\dim(NC(R))$ may be change after passing to the completion. This is the reason we use the notion of the polynomial type in this paper.

We next recall the object of the present paper.

Definition 2.6. Let $\mathfrak{q}$ be a parameter ideal of M. The *index of reducibility* of $\mathfrak{q}$ on M is the number of irreducibility components appear in an irredundant irreducible decomposition of $\mathfrak{q}M$, and denoted by $\mathcal{N}_R(\mathfrak{q}, M)$.

Remark 2.7. (i) It is well known that $\mathcal{N}_R(\mathfrak{q}, M) = \dim_{R/\mathfrak{m}} \mathrm{Soc}(M/\mathfrak{q}M)$, where $\mathrm{Soc}(N) = 0 :_N \mathfrak{m} \cong \mathrm{Hom}(R/\mathfrak{m}, N)$ for an arbitrary R-module N.

(ii) If M is Cohen-Macaulay i.e. $p(M) = -\infty$, then $\mathcal{N}_R(\mathfrak{q}, M) = \dim_{R/\mathfrak{m}} \mathrm{Soc}(H^d_{\mathfrak{m}}(M))$ for all parameter ideals $\mathfrak{q}$.

(iii) If M is generalized Cohen-Macaulay i.e. $p(M) = 0$, then Goto and Suzuki proved that

$$\mathcal{N}_R(\mathfrak{q}, M) \le \sum_{i=0}^{d-1} \binom{d}{i} \ell(H^i_{\mathfrak{m}}(M)) + \dim_{R/\mathfrak{m}} \mathrm{Soc}(H^d_{\mathfrak{m}}(M))$$

for all paprameter ideals $\mathfrak{q}$ of M (cf. [12, Theorem 2.1]). Furthermore, let n_0 be a positive integer such that $\mathfrak{m}^{n_0} H^i_{\mathfrak{m}}(M) = 0$ for all $i = 0, ..., d-1$. In [7, Corollary 4.3] N.T. Cuong and the author showed that for all parameter ideal $\mathfrak{q}$ contained in $\mathfrak{m}^{2n_0}$ we have

$$\mathcal{N}_R(\mathfrak{q}, M) = \sum_{i=0}^{d} \binom{d}{i} \dim_{R/\mathfrak{m}} \mathrm{Soc}(H^i_{\mathfrak{m}}(M)).$$

3 Modules of dimension one

Notice that $p(M)$ and $\mathcal{N}_R(\mathfrak{q}, M)$ do not change after passing to the $\mathfrak{m}$-adic completion. Therefore, in the rest of this paper we always assume that $(R, \mathfrak{m})$ is a complete ring. In this paper we consider the boundness of $\mathcal{N}_R(\mathfrak{q}, M)$ provided $p(M) \leq 1$. In this case Theorem 2.3 implies that $H^i_{\mathfrak{m}}(M)$ is Artinian with $\dim R/\mathrm{Ann}H^i_{\mathfrak{m}}(M) \leq 1$ for all $i = 0, ..., d-1$. Hence the Matlis' dual of $H^i_{\mathfrak{m}}(M)$ is a Noetherian module of dimension at most one for all $i = 0, ..., d-1$. The minimal number of generators of a module N will be denoted by $v(N)$. The key role in our proof of the main result is the following interesting result of local ring of dimension one (see [15, Chapter 3]).

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a local ring of dimension one. Then the minimal number of generators of ideals of R is bounded above by an invariant independent of the choice of ideals i.e there is a positive integer c such that $v(I) = \ell(I/\mathfrak{m}I) \leq c$ for all ideal I.*

Goto and Suzuki in [12, Theorem 3.1] extended above result for modules as follows.

Lemma 3.2. *Let M be a finitely generated R-module of dimension one. Then there is a positive integer c such that $v(N) = \ell(N/\mathfrak{m}N) \leq c$ for all submodule N of M.*

Notation 3.3. Let M be a finitely generated R-module. We define

$$c(M) = \sup_N \{v(N) \mid N \subseteq M\}.$$

Remark 3.4. (i) By Lemma 3.2 we have if $d \leq 1$, then $c(M)$ is a positive integer. Moreover if $d = 0$ then $c(M) \leq \ell(M)$.

(ii) If $d \geq 2$ since $v(\mathfrak{m}^n M)$ is a polynomial of degree $d-1$ when $n \gg 0$, then $c(M) = \infty$.

We present some properties of the invariant $c(M)$.

Proposition 3.5. *Let M be a finitely generated R-module of dimension $d \leq 1$. Then*

$$c(M) = \sup_N \{\ell(N :_M \mathfrak{m}/N) \mid N \subseteq M\}.$$

Proof. The assertion follows form the facts

$$\ell(N :_M \mathfrak{m}/N) \leq v(N :_M \mathfrak{m}),$$

and

$$v(N) \leq \ell((\mathfrak{m}N :_M \mathfrak{m})/\mathfrak{m}N).$$

□

Proposition 3.6. *We consider the following short exact sequence of finitely generated R-modules of dimension at most one*

$$0 \to M_1 \to M \to M_2 \to 0.$$

Then

(i) $c(M_1) \leq c(M)$ *and* $c(M_2) \leq c(M)$.

(ii) $c(M) \leq c(M_1) + c(M_2)$.

Proof. (i) immediately follows from the definition of $c(M)$.
(ii) Let N be a submodule of M such that $v(N) = c(M)$. There are submodules N_1 and N_2 of M_1 and M_2, respectively, such that

$$0 \to N_1 \to N \to N_2 \to 0$$

is a short exact sequence. Then

$$c(M) = v(N) \leq v(N_1) + v(N_2) \leq c(M_1) + c(M_2).$$

□

4 The main result

Recalling that we always assume that $(R, \mathfrak{m})$ be a complete Notherian local ring. Let $E(R/\mathfrak{m})$ be the injective hull of R-module $R/\mathfrak{m}$. Let A be an Artinian R-module. We have the Matlis' dual of A, $N = \mathrm{Hom}(A, E(R/\mathfrak{m}))$, is Noetherian and $\mathrm{Ann}A = \mathrm{Ann}N$. In this section we say an Artinian R-module A of dimension t if its dual is a Noetherian module of dimension t, i.e. $\dim R/\mathrm{Ann}A = t$. It is well known that $H^i_{\mathfrak{m}}(M)$ is Artinian for all $i \geq 0$ (see [2, Chapter 7]). Theorem 2.3 claims that a finitely generated R-module M of dimension d and $p(M) \leq 1$ if and only if $\dim H^i_{\mathfrak{m}}(M) \leq 1$ for all $i = 0, ..., d-1$. For the study of dimension of an Artinian module on a general Noetherian local ring see [6]. We need the following lemma.

Lemma 4.1 ([2], Lemma 10.2.16)**.** *Let E, F, I be R-module such that E is finitely generated and I is injective. Then*

$$\mathrm{Hom}(\mathrm{Hom}(E, F), I) \cong E \otimes \mathrm{Hom}(F, I).$$

For an Artinian R-module A we set $r(A) := \sup\{\ell(B :_A \mathfrak{m}/B) \mid B \subseteq A\}$.

Corollary 4.2. *Let A be an Artinian R-module of dimension at most one. Let $N = \mathrm{Hom}(A, E(R/\mathfrak{m}))$. Then $r(A) = c(N)$.*

Proof. For each submodule B of A set $L = \mathrm{Hom}(A/B, E(R/\mathfrak{m}))$, then L is a submodule of N. By Lemma 4.1 we have

$$\mathrm{Hom}(\mathrm{Hom}(R/\mathfrak{m}, A/B), E(R/\mathfrak{m})) \cong R/\mathfrak{m} \otimes \mathrm{Hom}(A/B, E(R/\mathfrak{m})).$$

Hence $\ell(B :_A \mathfrak{m}/B) = v(L) \leq c(N)$. Thus $r(A) \leq c(N)$. Conversely, let L be a submodule of N such that $v(L) = c(N)$. Let $B = \mathrm{Hom}(N/L, E(R/\mathfrak{m}))$. We have B is a submodule of A. By duality we have $N/L \cong \mathrm{Hom}(B, E(R/\mathfrak{m}))$ so $L \cong \mathrm{Hom}(A/B, E(R/\mathfrak{m}))$. As above we have $\ell(B :_A \mathfrak{m}/B) = v(L) = c(N)$, so $r(A) \geq c(N)$. ∎

The next result immediately follows from Corollary 4.2 and Proposition 3.6.

Corollary 4.3. *We consider the following short exact sequence of Artinian R-modules of dimension at most one*

$$0 \to A_1 \to A \to A_2 \to 0.$$

Then

(i) $r(A_1) \leq r(A)$ *and* $r(A_2) \leq r(A)$.

(ii) $r(A) \leq r(A_1) + r(A_2)$.

Recalling that a sequence of elements $x_1, ..., x_k$ is called a *filter regular sequence* of M if $\mathrm{Supp}\,((x_1, ..., x_{i-1})M : x_i)/(x_1, ..., x_{i-1})M \subseteq \{\mathfrak{m}\}$ for all $i = 1, ..., k$.

Proposition 4.4. *Let M be a finitely generated R-module of dimension d and $p(M) \leq 1$. Then for every filter regular sequence $x_1, ..., x_k$, $k \leq d$, of M we have*

$$r(H^j_{\mathfrak{m}}(M/(x_1, ..., x_k)M)) \leq \sum_{i=j}^{j+k} \binom{k}{i-j} r(H^i_{\mathfrak{m}}(M))$$

for all $j < d-k$, and

$$\dim \mathrm{Soc}(H^{d-k}_{\mathfrak{m}}(M/(x_1,...,x_k)M)) \leq \sum_{i=d-k}^{d-1} \binom{k}{k+i-d} r(H^i_{\mathfrak{m}}(M)) + \dim \mathrm{Soc}(H^d_{\mathfrak{m}}(M)).$$

Proof. Induction on k, the case $k=0$ is trivial. If $k=1$, the short exact sequence

$$0 \to M/0 :_M x_1 \to M \to M/x_1M \to 0$$

induces the following exact sequence

$$H^j_{\mathfrak{m}}(M) \to H^j_{\mathfrak{m}}(M/x_1M) \to H^{j+1}_{\mathfrak{m}}(M/0 :_M x_1)$$

for all $j < d-1$. Since $\ell(0 :_M x_1) < \infty$ we have $H^{j+1}_{\mathfrak{m}}(M/0 :_M x_1) \cong H^{j+1}_{\mathfrak{m}}(M)$ for all $j \geq 0$. Hence $\mathrm{Ann} H^j_{\mathfrak{m}}(M/x_1M) \supseteq \mathrm{Ann} H^j_{\mathfrak{m}}(M) \mathrm{Ann} H^{j+1}_{\mathfrak{m}}(M)$ for all $j < d-1$. Therefore $\dim R/\mathrm{Ann} H^j_{\mathfrak{m}}(M/x_1M) \leq 1$ for all $j = 0,...,d-2$, so $p(M/x_1M) \leq 1$. By Corollary 4.3 it is clear that

$$r(H^j_{\mathfrak{m}}(M/x_1M)) \leq r(H^j_{\mathfrak{m}}(M)) + r(H^{j+1}_{\mathfrak{m}}(M))$$

for all $j < d-1$. On the other hand we have the following exact sequence

$$H^{d-1}_{\mathfrak{m}}(M) \to H^{d-1}_{\mathfrak{m}}(M/x_1M) \to H^d_{\mathfrak{m}}(M) \xrightarrow{x} H^d_{\mathfrak{m}}(M).$$

Thus we have the short exact sequence

$$0 \to A \to H^{d-1}_{\mathfrak{m}}(M/x_1M) \to 0 :_{H^d_{\mathfrak{m}}(M)} x_1 \to 0,$$

where A is a quotient of $H^{d-1}_{\mathfrak{m}}(M)$. By applying the functor $\mathrm{Hom}(R/\mathfrak{m}, \bullet)$ to the above short exact sequence we get the following exact sequence

$$0 \to 0 :_A \mathfrak{m} \to 0 :_{H^{d-1}_{\mathfrak{m}}(M/x_1M)} \mathfrak{m} \to 0 :_{H^d_{\mathfrak{m}}(M)} \mathfrak{m}.$$

Therefore

$$\begin{aligned} \dim \mathrm{Soc}(H^{d-1}_{\mathfrak{m}}(M/x_1M)) &\leq \dim \mathrm{Soc}(A) + \dim \mathrm{Soc}(H^d_{\mathfrak{m}}(M)) \\ &\leq r(A) + \dim \mathrm{Soc}(H^d_{\mathfrak{m}}(M)) \\ &\leq r(H^{d-1}_{\mathfrak{m}}(M)) + \dim \mathrm{Soc}(H^d_{\mathfrak{m}}(M)). \end{aligned}$$

So the assertion holds true if $k=1$. For $k>1$ by induction we have

$$\begin{aligned} r(H^j_{\mathfrak{m}}(M/(x_1,...,x_k)M)) &\leq r(H^j_{\mathfrak{m}}(M/(x_1,...,x_{k-1})M)) + r(H^{j+1}_{\mathfrak{m}}(M/(x_1,...,x_{k-1})M)) \\ &\leq \sum_{i=j}^{k+j-1} \binom{k-1}{i-j} r(H^i_{\mathfrak{m}}(M)) + \sum_{i=j+1}^{k+j} \binom{k-1}{i-j-1} r(H^i_{\mathfrak{m}}(M)) \\ &= \sum_{i=j}^{j+k} \binom{k}{i-j} r(H^i_{\mathfrak{m}}(M)) \end{aligned}$$

for all $j < d-k$. Moreover, we have

$$\begin{aligned} &\dim \mathrm{Soc}(H^{d-k}_{\mathfrak{m}}(M/(x_1,...,x_k)M)) \\ &\quad\leq r(H^{d-k}_{\mathfrak{m}}(M/(x_1,...,x_{k-1})M)) + \dim \mathrm{Soc}(H^{d-k+1}_{\mathfrak{m}}(M/(x_1,...,x_{k-1})M)) \\ &\quad\leq \sum_{k=d-k}^{d-1} \binom{k-1}{k+i-d} r(H^i_{\mathfrak{m}}(M)) + \sum_{i=d-k+1}^{d-1} \binom{k-1}{k+i-d-1} r(H^i_{\mathfrak{m}}(M)) + \dim \mathrm{Soc}(H^d_{\mathfrak{m}}(M)) \\ &\quad= \sum_{i=d-k}^{d-1} \binom{k}{k+i-d} r(H^i_{\mathfrak{m}}(M)) + \dim \mathrm{Soc}(H^d_{\mathfrak{m}}(M)). \end{aligned}$$

The proof is complete. ∎

Remark 4.5. It should be noted that for every parameter ideal $\mathfrak{q} = (x_1, ..., x_d)$ of M we can choose a system of parameters $\underline{y} = y_1, ..., y_d$ which is a filter regular sequence such that $\mathfrak{q} = (y_1, ..., y_d)$. Indeed, by the prime avoidance theorem we can choose an element $y_1 \in \mathfrak{q} \setminus \mathfrak{m}\mathfrak{q}$ and $y_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}M$ and $\mathfrak{p} \neq \mathfrak{m}$. Therefore y_1 is both a parameter element and a filter regular element of M. For $i = 2, ..., d$, by applying the prime avoidance theorem again there exists an element $y_i \in \mathfrak{q} \setminus \big(\mathfrak{m}\mathfrak{q} \cup (y_1, ..., y_{i-1})\big)$ and $y_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}M/(y_1, ..., y_{i-1})M$ and $\mathfrak{p} \neq \mathfrak{m}$. Thus we have a system of parameters $\underline{y} = y_1, ..., y_d$ which is also a filter regular sequence of M. The claim $\mathfrak{q} = (y_1, ..., y_d)$ is easy by the Nakayama lemma.

Applying for $k = d$ in Proposition 4.4 and using Remark 4.5 we have the main result of this paper as follows.

Theorem 4.6. *Let M be a finitely generated R-module of dimension d and $p(M) \leq 1$. Then the index of reducibility of parameter ideal $\mathfrak{q}$ of M is bounded above by a invariant independent of the choice of $\mathfrak{q}$. Namely*

$$\mathcal{N}_R(\mathfrak{q}, M) \leq \sum_{i=0}^{d-1} \binom{d}{i} r(H^i_{\mathfrak{m}}(M)) + \dim \mathrm{Soc}(H^d_{\mathfrak{m}}(M))$$

for all parameter ideals $\mathfrak{q}$ of M.

Notice that in the case M is generalized Cohen-Macaulay our bound is a sharp of the Goto-Suzuki one since $r(H^i_{\mathfrak{m}}(M)) \leq \ell(H^i_{\mathfrak{m}}(M))$ for all i. An R-module M is called *unmixed* if $\dim R/\mathfrak{p} = \dim M$ for all $\mathfrak{p} \in \mathrm{Ass}M$. It is not difficult to see that if M is an unmixed module of dimension three, then $p(M) \leq 1$ (see [3, Theorem 8.1.1]). The following is a generalization of [12, Corollary 3.7] for modules.

Corollary 4.7. *Let M is an unmixed module of dimension three. Then the index of reducibility of parameter ideal $\mathfrak{q}$ of M is bounded above by a invariant independent of the choice of $\mathfrak{q}$.*

If $(R, \mathfrak{m})$ is a local ring of dimension three, then $\mathcal{N}_R(\mathfrak{q}, R)$ is bounded above for all parameter ideals $\mathfrak{q}$ of R (cf. [12, Theorem 3.8]). Thus the class of modules for which the index of reducibility of parameter ideals is bounded is strictly larger than the class of modules of the polynomial type at most one. In [12, Example 3.9] Goto and Suzuki also constructed a ring of dimension four and the index of reducibility of parameter ideals are not bounded above. Notice that the ring of Goto and Suzuki has the polynomial type three. Therefore it is natural to raise the following question.

Question 4.8. Is it true that $\mathcal{N}_R(\mathfrak{q}, M)$ is bounded above for all parameter ideals $\mathfrak{q}$ of M if and only if $p(M) \leq 2$.

Acknowledgements: The author is grateful to Professor Nguyen Tu Cuong for his valuable comments on the first draft of this paper. This paper has been written while the author is visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM), Hanoi, Vietnam. He would like to thank VIASM for their support and hospitality.

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