A Remark on the Finiteness Dimension [1]

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Dedicated to Professor Nguyen Tu Cuong on the occasion of his sixtieth birthday

Abstract

This note is the main part of my report at the 33rd symposium on Commutative Algebra in Japan. The rest of my report can be seen in [14]. Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R and M a finitely generated R-module. The finiteness dimension of M relative to $\mathfrak{a}$ is defined by

$$f_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 \, : \, H^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\},$$

where $H^i_{\mathfrak{a}}(M)$ is the i-th local cohomology with respect to $\mathfrak{a}$. The aim of this paper is to show that if $x_1, ..., x_t$ is an $\mathfrak{a}$-filter regular sequence of M with $t \leq f_{\mathfrak{a}}(M)$, then the set

$$\bigcup_{n_1,...,n_t \in \mathbb{N}} \operatorname{Ass} M/(x_1^{n_1}, ..., x_t^{n_t})M$$

is finite.

1 Introduction

Throughout this paper, let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R and M a finitely generated R-module. For basic facts about local cohomology refer to [2]. We use $\mathbb{N}_0$ (resp. $\mathbb{N}$) to denote the set of non-negative (resp. positive) integers.

Local cohomology was introduced by A. Grothendieck. In general, the i-th local cohomology of M with respect to $\mathfrak{a}$, $H^i_{\mathfrak{a}}(M)$, may not be finitely generated. An important problem in Commutative Algebra is to find certain finiteness properties of local cohomology. In [4], C. Huneke raised the following conjecture: Is the number of associated prime ideals of a local cohomology module $H^i_{\mathfrak{a}}(M)$ always finite? This question has received much attention in the case when $M = R$ is a regular ring (cf. [5], [10], [16]). Although A.K. Singh in [15] gave the first counterexample to Huneke's conjecture, it has positive answer in many cases. For a given positive integer t, $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$ is finite if either $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < t$ (cf. [1], [8]) or $\operatorname{Supp} H^i_{\mathfrak{a}}(M)$ is finite for all $i < t$ (cf. [8]). Combining these results, the author in [14] showed that $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$ is finite if for each $i < t$ either $H^i_{\mathfrak{a}}(M)$ is finitely generated or $\operatorname{Supp} H^i_{\mathfrak{a}}(M)$ is a finite set.

As mentioned above, if t is the least integer such that $H^t_{\mathfrak{a}}(M)$ is not finitely generated, then $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$ is finite. Such integer is called the *finiteness dimension, denoted by $f_{\mathfrak{a}}(M)$, of M relative to* $\mathfrak{a}$ (see, [2, Chapter 9]). The purpose of this paper is to show that the finiteness dimension provides a stronger result about the finiteness of certain sets of associated primes. Namely, let $x_1, ..., x_t$ be an $\mathfrak{a}$-filter regular sequence of M with $t \leq f_{\mathfrak{a}}(M)$, i.e. $\operatorname{Supp}((x_1, ..., x_{i-1})M : x_i)/(x_1, ..., x_{i-1})M \subseteq V(\mathfrak{a})$ for all $i = 1, ..., t$, where $V(\mathfrak{a})$ denotes the set of prime ideals containing $\mathfrak{a}$. Then the set

$$\bigcup_{n_1,...,n_t \in \mathbb{N}} \operatorname{Ass} M/(x_1^{n_1}, ..., x_t^{n_t})M$$

is finite.

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2 The main result

Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R, and M a finitely generated R-module. We begin by recalling some facts about the finiteness dimension of M relative to $\mathfrak{a}$.

Definition 2.1. (i) The *finiteness dimension of M relative to $\mathfrak{a}$* is defined by

$$f_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 \,:\, H^i_{\mathfrak{a}}(M) \,\text{is not finitely generated}\},$$

with the usual convention that the infimum of the empty set of integers is ∞.

(ii) The $\mathfrak{a}$-*minimum* $\mathfrak{a}$-*adjusted depth of M* is defined by

$$\lambda_{\mathfrak{a}}(M) = \inf\{\operatorname{depth} M_{\mathfrak{p}} + \operatorname{ht}(\mathfrak{a}+\mathfrak{p})/\mathfrak{p} \,:\, \mathfrak{p} \in \operatorname{Supp}(M) \setminus V(\mathfrak{a})\},$$

with the convention that $\operatorname{ht}(\mathfrak{a}+\mathfrak{p})/\mathfrak{p} = \infty$ if $\mathfrak{a}+\mathfrak{p} = R$.

Remark 2.2. (i) $f_{\mathfrak{a}}(M) \in \mathbb{N}_0$ provided $\mathfrak{a}M \neq M$ and M is not $\mathfrak{a}$-torsion.

(ii) $f_{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N} \,:\, \mathfrak{a}^n H^i_{\mathfrak{a}}(M) \neq 0 \,\text{for all}\, n \in \mathbb{N}\}$. Therefore there exists a positive integer n_0 such that $\mathfrak{a}^{n_0} H^i_{\mathfrak{a}}(M) = 0$ for all $i < f_{\mathfrak{a}}(M)$.

(iii) $f_{\mathfrak{a}}(M) \leq \lambda_{\mathfrak{a}}(M)$ and the equality holds when R is universally catenary and all the formal fibres of all its localizations are Cohen-Macaulay rings (see, [2, 9.6.7]).

We next recall the notion of $\mathfrak{a}$-filter regular sequence of M and its relation with local cohomology.

Definition 2.3. We say a sequence $x_1, ..., x_t$ of elements contained in $\mathfrak{a}$ is an $\mathfrak{a}$-*filter regular sequence of M* if

$$\operatorname{Supp}\left((x_1, ..., x_{i-1})M : x_i)/(x_1, ..., x_{i-1})M \subseteq V(\mathfrak{a})\right.$$

for all $i = 1, ..., t$, where $V(\mathfrak{a})$ denotes the set of prime ideals containing $\mathfrak{a}$.

Remark 2.4. Let $x_1, ..., x_t$ be an $\mathfrak{a}$-filter regular sequence of M. Then

(i) For all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus V(\mathfrak{a})$, $\frac{x_1}{1}, \cdots, \frac{x_t}{1}$ is a poor $M_{\mathfrak{p}}$-sequence i.e. for each $i = 2, ..., t$, the element x_i is a non-zerodivisor on $M/(x_1, ..., x_{t-1})M$ (cf. [12, Proposition 2.2]).

(ii) $x_1^{n_1}, ..., x_t^{n_t}$ is an $\mathfrak{a}$-filter regular sequence of M for all $n_1, ..., n_t \in \mathbb{N}$, moreover

$$\operatorname{Ass}(M/(x_1^{n_1}, ..., x_t^{n_t})M) \setminus V(\mathfrak{a}) = \operatorname{Ass}(M/(x_1, ..., x_t)M) \setminus V(\mathfrak{a}).$$

(iii) By [12, Proposition 3.4] we have $H^t_{\mathfrak{a}}(M) \cong H^0_{\mathfrak{a}}(H^t_{(x_1,...,x_t)}(M))$. Combining with the well-known fact that $H^t_{(x_1,...,x_t)}(M) \cong \lim_{\rightarrow} M/(x_1^{n_1}, ..., x_t^{n_t})M$, it follows that

$$\operatorname{Ass} H^t_{\mathfrak{a}}(M) \subseteq \bigcup_{n_1,...,n_t \in \mathbb{N}} \operatorname{Ass} M/(x_1^{n_1}, ..., x_t^{n_t})M.$$

Proof of (ii). By [12, Proposition 2.2] we have $x_1^{n_1}, ..., x_t^{n_t}$ is an $\mathfrak{a}$-filter regular sequence of M for all $n_1, ..., n_t \in \mathbb{N}$. Let $\mathfrak{p} \in \operatorname{Ass}(M/(x_1, ..., x_t)M) \setminus V(\mathfrak{a})$. By localization at $\mathfrak{p}$ we have $\mathfrak{p}R_{\mathfrak{p}}\operatorname{Ass}(M_{\mathfrak{p}}/(\frac{x_1}{1}, \cdots, \frac{x_t}{1})M_{\mathfrak{p}})$ and $\frac{x_1}{1}, \cdots, \frac{x_t}{1}$ is an $M_{\mathfrak{p}}$-sequence. The assertion now follows from the fact that

$$\operatorname{Ass}(M/(x_1^{n_1}, ..., x_t^{n_t})M) = \operatorname{Ass}(M/(x_1, ..., x_t)M)$$

for all $n_1, ..., n_t \in \mathbb{N}$ provided $x_1, ..., x_t$ is an M-sequence. □

Recently, N.T. Cuong and the author proved the following splitting theorem (cf. [3]) whose consequence plays a key role in this paper.

Theorem 2.5 ([3], Theorem 1.1). *Let M be a finitely generated module over a Noetherian ring R and $\mathfrak{a}$ an ideal of R. Let t and n_0 be positive integers such that $\mathfrak{a}^{n_0}H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Then, for all $\mathfrak{a}$-filter regular element $x \in \mathfrak{a}^{2n_0}$ of M, it holds that*

$$H^i_{\mathfrak{a}}(M/xM) \cong H^i_{\mathfrak{a}}(M) \oplus H^{i+1}_{\mathfrak{a}}(M),$$

for all $i < t - 1$, and

$$0 :_{H^{t-1}_{\mathfrak{a}}(M/xM)} \mathfrak{a}^{n_0} \cong H^{t-1}_{\mathfrak{a}}(M) \oplus 0 :_{H^t_{\mathfrak{a}}(M)} \mathfrak{a}^{n_0}.$$

Corollary 2.6 ([3], Corollary 4.4). *Let M be a finitely generated R-module and $\mathfrak{a}$ an ideal of R. Let t and n_0 be positive integers such that $\mathfrak{a}^{n_0}H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Then for every $\mathfrak{a}$-filter regular sequence $x_1, ..., x_t$ of M contained in $\mathfrak{a}^{2n_0}$, we have*

$$\bigcup_{i=0}^{j} \operatorname{Ass} H^i_{\mathfrak{a}}(M) = \operatorname{Ass}(M/(x_1, ..., x_j)M) \bigcap V(\mathfrak{a}),$$

for all $j = 1, \ldots, t$. In particular, $H^t_{\mathfrak{a}}(M)$ has only finitely many associated primes.

Corollary 2.6 implies that $\cup_{n\in\mathbb{N}}\operatorname{Ass} M/(x_1^n, ..., x_t^n)M$ is finite for every $\mathfrak{a}$-filter regular sequence $x_1, ..., x_t$ of M with $t \leq f_{\mathfrak{a}}(M)$. In order to prove the main result we need some preliminary lemmas. The author is grateful to K. Khashyarmanesh for information that the following is a sharp of [7, Lemma 2.1].

Lemma 2.7. *Let M be a finitely generated R-module and $\mathfrak{a}$ an ideal of R. Let t and n_0 be positive integers such that $\mathfrak{a}^{n_0}H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Then for every $\mathfrak{a}$-filter regular sequence $x_1, ..., x_t$ of M, we have $\mathfrak{a}^{2^j n_0}H^i_{\mathfrak{a}}(M/(x_1, ..., x_j)M) = 0$ for all $0 \leq j \leq t-1$ and $i < t - j$.*

Proof. The case $j = 0$ is trivial and by induction it is sufficient to show the assertion in the case $j = 1 < t$. The short exact sequence

$$0 \longrightarrow M/(0 :_M x_1) \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

induces the exact sequence

$$\cdots \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(M/x_1M) \longrightarrow H^{i+1}_{\mathfrak{a}}(M/(0 :_M x_1)) \longrightarrow \cdots.$$

Notice that $0 :_M x_1$ is $\mathfrak{a}$-torsion, hence $H^{i+1}_{\mathfrak{a}}(M/(0 :_M x_1)) \cong H^{i+1}_{\mathfrak{a}}(M)$ for all $i \geq 0$. Thus $\mathfrak{a}^{n_0}H^{i+1}_{\mathfrak{a}}(M/(0 :_M x_1)) = 0$ for all $i < t - 1$. The assertion is now clear. □

Proposition 2.8. *Let M be a finitely generated R-module and $\mathfrak{a}$ an ideal of R. Let t and n_0 be positive integers such that $\mathfrak{a}^{n_0}H^i_{\mathfrak{a}}(M) = 0$ for all $i < t$. Let $x_1, ..., x_t$ be an $\mathfrak{a}$-filter regular sequence of M and $j < t$ a non-negative integer. For all $n_1, ..., n_t \in \mathbb{N}$ such that $n_i \geq 2^t n_0$ for all $j + 1 \leq i \leq t$, we have*

$$\operatorname{Ass} M/(x_1^{n_1}, ..., x_t^{n_t})M = \operatorname{Ass} M/(x_1^{n_1}, ..., x_j^{n_j}, x_{j+1}^{2^t n_0}, ..., x_t^{2^t n_0})M.$$

Proof. By Remark 2.4 (ii) we have

$$\operatorname{Ass}(M/(x_1^{n_1}, ..., x_t^{n_t})M) \setminus V(\mathfrak{a}) = \operatorname{Ass}(M/(x_1^{n_1}, ..., x_j^{n_j}, x_{j+1}^{2^t n_0}, ..., x_t^{2^t n_0})M) \setminus V(\mathfrak{a}).$$

On the other hand $\mathfrak{a}^{2^j n_0} H^i_{\mathfrak{a}}(M/(x_1^{n_1},...,x_j^{n_j})M) = 0$ for all $i < t-j$ by Lemma 2.7, and Corollary 2.6 implies that

$$\begin{aligned}\operatorname{Ass}(M/(x_1^{n_1},...,x_t^{n_t})M)\bigcap V(\mathfrak{a}) &= \bigcup_{i=0}^{t-j} \operatorname{Ass} H^i_{\mathfrak{a}}(M/(x_1^{n_1},...,x_j^{n_j})M) \\ &= \operatorname{Ass}(M/(x_1^{n_1},...,x_j^{n_j},x_{j+1}^{2^t n_0},...,x_t^{2^t n_0})M)\bigcap V(\mathfrak{a}).\end{aligned}$$

The proof is complete. □

Lemma 2.9. *Let $(R,\mathfrak{m})$ be a local ring. Let $x_1,...,x_t$ be an $\mathfrak{a}$-filter regular sequence of M such that $t \le \lambda_{\mathfrak{a}}(M)$, the $\mathfrak{a}$-minimum $\mathfrak{a}$-adjusted depth of M. Then $x_1,...,x_t$ is an $\mathfrak{a}$-filter regular sequence of M in any order.*

Proof. It is sufficient to show the assertion in the case $t = 2 \le \lambda_{\mathfrak{a}}(M)$. Moreover we only need to prove that x_2 is an $\mathfrak{a}$-filter regular element of M (see [6, Theorem 117]). Indeed, let $\mathfrak{p} \in \operatorname{Ass} M \backslash V(\mathfrak{a})$. Then $\operatorname{ht}(\mathfrak{a}+\mathfrak{p})/\mathfrak{p} \ge 2$ by the definition of $\lambda_{\mathfrak{a}}(M)$. Thus there exists $\mathfrak{q} \in \operatorname{Spec}(R) \setminus V(\mathfrak{a})$ such that $\mathfrak{q}$ is a minimal prime ideal of $(x_1)+\mathfrak{p}$. By localization at $\mathfrak{q}$ we have $\frac{x_1}{1}$ is a $M_{\mathfrak{q}}$-regular element. Hence $\mathfrak{q}R_{\mathfrak{q}} \in \operatorname{Ass}(M/x_1M)_{\mathfrak{q}}$ since $\operatorname{ht}(\mathfrak{q}R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) = 1$ and $\mathfrak{p}R_{\mathfrak{q}} \in \operatorname{Ass} M_{\mathfrak{q}}$. Thus $\mathfrak{q} \in \operatorname{Ass} M/x_1M$. Hence $x_2 \notin \mathfrak{q}$ because x_2 is an $\mathfrak{a}$-filter regular element of M/x_1M. Therefore $x_2 \notin \mathfrak{p}$ and so x_2 is an $\mathfrak{a}$-filter regular element of M. □

We now give the main result of this paper.

Theorem 2.10. *Let M be a finitely generated R-module, and $\mathfrak{a}$ an ideal of R. Let t be a positive integer such that $t \le f_{\mathfrak{a}}(M)$, the finiteness dimension of M relative to $\mathfrak{a}$, and $x_1,...,x_t$ an $\mathfrak{a}$-filter regular sequence of M. Then the set*

$$\bigcup_{n_1,...,n_t \in \mathbb{N}} \operatorname{Ass} M/(x_1^{n_1},...,x_t^{n_t})M$$

is finite.

Proof. Let n_0 be a positive integer such that $\mathfrak{a}^{n_0} H^i_{\mathfrak{a}}(M) = 0$ for all $i < f_{\mathfrak{a}}(M)$. For each $(n_1,...,n_t) \in \mathbb{N}^t$ we consider a t-tuple of positive integers $(m_1,...,m_t) \in \mathbb{N}^t$ such that $m_i = n_i$ if $n_i < 2^t n_0$, and $m_i = 2^t n_0$ if $n_i \ge 2^t n_0$. We have that $\mathfrak{p} \in \operatorname{Ass} M/(x_1^{n_1},...,x_t^{n_t})M$ iff $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass} M_{\mathfrak{p}}/(x_1^{n_1},...,x_t^{n_t})M_{\mathfrak{p}}$. By Lemma 2.9 and a change of the order of the x_i, if necessary, we can assume that $n_i < 2^t n_0$ for all $i \le j$, and $n_i \ge 2^t n_0$ for all $j+1 \le i \le t$, for some $j \le t$. Now, Proposition 2.8 implies that $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass} M_{\mathfrak{p}}/(x_1^{n_1},...,x_t^{n_t})M_{\mathfrak{p}}$ iff $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass} M_{\mathfrak{p}}/(x_1^{m_1},...,x_t^{m_t})M_{\mathfrak{p}}$. Therefore

$$\operatorname{Ass} M/(x_1^{n_1},...,x_t^{n_t})M = \operatorname{Ass} M/(x_1^{m_1},...,x_t^{m_t})M.$$

Hence

$$\bigcup_{n_1,...,n_t \in \mathbb{N}} \operatorname{Ass} M/(x_1^{n_1},...,x_t^{n_t})M = \bigcup_{1 \le m_1,...,m_t \le 2^t n_0} \operatorname{Ass} M/(x_1^{m_1},...,x_t^{m_t})M$$

is a finite set. □

It should be noted that L.T. Nhan in [13, Theorem 3.1] proved a similar result for generalized regular sequences of M. We recall that in a local ring $(R,\mathfrak{m})$ a sequence $x_1,...,x_t$ of elements is said to be a *generalized regular sequence of* M if $x_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M/(x_1,...,x_{i-1})M$ satisfying $\dim R/\mathfrak{p} > 1$, for all $i = 1,...,t$.

Question 2.11. Notice that $H^i_{\mathfrak{a}}(M) = \lim_{\to} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$, by virtue of Theorem 2.10 it raises the following natural questions.

(i) Is $\cup_n \operatorname{Ass}\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ finite for all $i \leq f_{\mathfrak{a}}(M)$?

(ii) Is

$$\bigcup_{n_1,\dots,n_t\in\mathbb{N}} \operatorname{Ass}\operatorname{Ext}^i_R(R/(x_1^{n_1},...,x_t^{n_t}), M)$$

finite for all $\mathfrak{a}$-filter regular sequence $x_1, ..., x_t$ of M and $i \leq t \leq f_{\mathfrak{a}}(M)$?

If M is an $\mathfrak{a}$-torsion module, then $f_{\mathfrak{a}}(M) = \infty$. The following is a special case of Question 2.11(i).

Question 2.12. Is $\cup_n \operatorname{Ass}\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ finite for all i provided M is $\mathfrak{a}$-torsion?

In [11], L. Melkersson and Schenzel asked whether the sets $\operatorname{Ass}\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ become stable for sufficiently large n. This question is not true in general since $\cup_n \operatorname{Ass}\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ may be infinite. However, Khashyarmanesh and Salarian have proved that $\operatorname{Ass}\operatorname{Ext}^1_R(R/\mathfrak{a}^n, M)$ become stable for sufficiently large n (cf. [9, Corollary 2.3]). Thus, Melkersson-Schenzel's question and Question 2.11 (i) has an affirmative answer in the cases $f_{\mathfrak{a}}(M) \leq 1$. We may modify Melkersson-Schenzel's question as follows.

Question 2.13. *whether the sets* $\operatorname{Ass}\operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ *become stable for sufficiently large* n *and for all* $i \leq f_{\mathfrak{a}}(M)$*?*

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