Localization at countably infinitely many prime ideals and applications [1]

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Abstract

In this paper we present a technical lemma about localization at countably infinitely many prime ideals. We apply this lemma to get many results about the finiteness of associated prime ideals of local cohomology modules.

1 Introduction

In this paper, let R be a commutative Noetherian ring. Localization is one of the most important tools in Commutative algebra. Notice that for any multiplicative subset S of R, the canonical extension $R \to R_S$ is flat, and many problems in Commutative algebra have good behavior under flat extensions. For a set of finitely many prime ideals $\{\mathfrak{p}_1, ..., \mathfrak{p}_k\}$ with no containment relations, set $S = R \setminus \cup_{i=1}^k \mathfrak{p}_i$, we have R_S is a semilocal ring and $\mathrm{Max}(R_S) = \{\mathfrak{p}_1 R_S, ..., \mathfrak{p}_k R_S\}$. This fact follows from the well known prime avoidance lemma. This statement is false for countably infinitely many prime ideals $\{\mathfrak{p}_i\}_{i\geq 1}$. For example, let $R = \mathbb{Q}[X, Y]$ and $\{\mathfrak{p}_i\}_{i\in I}$ is the set of prime ideals of height one. Since R is UFD we have a prime ideal of height one is principal. Moreover R is a countable set, so the set $\{\mathfrak{p}_i\}_{i\in I}$ is countable. On the other hand every non-constant polynomial must be contained in a prime ideal of height one. Thus $S = R \setminus \cup_{i\in I}\mathfrak{p}_i = \mathbb{Q}$ and so $R_S = R$. This paper is devoted to the localization at countably infinitely many prime ideals after passing to a certain flat extension. Concretely, we prove the following result.

Lemma 1.1. *Let R be a commutative Noetherian ring and $\{\mathfrak{p}_i\}_{i\geq 1}$ a countable set of prime ideals of R with no containment relation. Consider the formal power series ring $R[[X]]$ and set $S = R[[X]] \setminus \cup_{i\geq 1}\mathfrak{p}_i R[[X]]$ and $T = R[[X]]_S$. Then $R \to T$ is a flat extension and $\mathrm{Max}(T) = \{\mathfrak{p}_i T\}_{i\geq 1}$.*

The above lemma will be proved in the next section. In Section 3 we apply Lemma 1.1 to get many results about the the finiteness of associated prime ideals of local cohomology modules. Among them, is the following:

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Theorem 1.2. *Let I be an ideal of R and M a finitely generated R-module. Then for every $i \geq 0$ the set $\{\mathfrak{p} \in \mathrm{Ass}_R H^i_I(M) \ : \ \mathrm{ht}(\mathfrak{p}/I) \leq 1\}$ is finite.*

Recall that, for any ideal I of R and any R-module M, the i^{th} local cohomology module of M with respect to I is defined as

$$H^i_I(M) = \varinjlim_{n \geq 1} \mathrm{Ext}^i_R(R/I^n, M).$$

We refer the reader to [2] or [4] for more details about local cohomology.

2 Localization at countably infinitely many prime ideals

We start this section with the well known result, countable prime avoidance lemma (see [9, Lemma 13.2]).

Lemma 2.1. *Let A be a Noetherian ring satisfying either of these conditions:*

(i) *A is a complete local ring.*

(ii) *There is an uncountable set of elements $\{\mu_\lambda\}_\lambda \in \Lambda$ such that $\mu_\lambda - \mu_\gamma$ is a unit of A for every $\lambda \neq \gamma$.*

Let $\{\mathfrak{p}_i\}_{i \geq 1}$ a countable set of prime ideals of A and I an ideal such that $I \subseteq \cup_{i \geq 1} \mathfrak{p}_i$. Then $I \subseteq \mathfrak{p}_i$ for some i.

The following technical lemma is the main result of this section.

Lemma 2.2. *Let R be a commutative Noetherian ring and $\{\mathfrak{p}_i\}_{i \geq 1}$ a countable set of prime ideals of R with no containment relation. Consider the formal power series ring $R[[X]]$ and set $S = R[[X]] \setminus \cup_{i \geq 1} \mathfrak{p}_i R[[X]]$ and $T = R[[X]]_S$. Then $R \to T$ is a flat extension and $\mathrm{Max}(T) = \{\mathfrak{p}_i T\}_{i \geq 1}$.*

Proof. It is clear that $R \to T$ is flat and $\mathfrak{p}_i T \in \mathrm{Spec}(T)$ for all $i \geq 1$. We prove that T satisfies the condition (ii) of Lemma 2.1. We consider the following subset of elements in $R[[X]]$

$$\mathcal{B} := \{\mu_\lambda = b_0 + b_1 X + \cdots + b_n X^n + \cdots \ : \ b_i = 0 \text{ or } 1 \text{ and } \mu_\lambda \neq 0\}.$$

Since $R[[X]]$ is a subring of T, so $\mathcal{B} \subseteq T$. It is clear that $\mathcal{B}$ is an uncountable set. For every $\mu_\lambda \neq \mu_\gamma$ pair of distinct elements of $\mathcal{B}$ we have

$$\mu_\lambda - \mu_\gamma = a_0 + a_1 X + \cdots + a_n X^n + \cdots$$

with $a_i = 0, 1$ or -1 and at least one $a_i \neq 0$. Let k be the least integer such that $a_k \neq 0$. We have

$$\mu_\lambda - \mu_\gamma = X^k(1 + a_{k+1}X + \cdots)$$

or

$$\mu_\lambda - \mu_\gamma = X^k(-1 + a_{k+1}X + \cdots).$$

We have both $1 + a_{k+1}X + \cdots$ and $-1 + a_{k+1}X + \cdots$ are units in $R[[X]]$ and so are in T. Since $X \notin \mathfrak{p}_i T$ for all $i \geq 1$ we have $X \in S$. Thus X is a unit in T. Therefore $\mu_\lambda - \mu_\gamma$ is a unit in T for every $\mu_\lambda \neq \mu_\gamma$. Hence T satisfies the countable prime avoidance lemma. Since $\cup_{i \geq 1} \mathfrak{p}_i T$ is the set of non-units of T, we have $I \subseteq \cup_{i \geq 1} \mathfrak{p}_i T$ for every proper ideal I of T. By the countable prime avoidance lemma we have $I \subseteq \mathfrak{p}_i T$ for some i. Therefore $\mathrm{Max}(T) = \{\mathfrak{p}_i T\}_{i \geq 1}$. The proof is complete. $\square$

3 Applications

In this section, let I be an ideal of R and M a finitely generated R-module. In general the i^{th} local cohomology module $H^i_I(M)$ is not finitely generated. Grothendieck asked the following question: Is $\mathrm{Hom}(R/I, H^i_I(M))$ finitely generated for all $i \geq 0$? The first counterexample was given by Hartshorne in [5]. In this paper he introduced the notion of I-cofinite modules. An R-module L is called I-cofinite if $\mathrm{Supp}(L) \subseteq V(I)$ and $\mathrm{Ext}^i_R(R/I, L)$ is finitely generated for all $i \geq 0$. Hartshorne proved that $H^i_I(M)$ is I-cofinite for all $i \geq 0$ if R is a complete regular local ring and I is a prime ideal of dimension one. Hartshorne's result was extended by many authors (see, [1], [3], [7], [13]). In [1, Theorem 1.1] Bahmanpour and Naghipour proved the following result (see also [12, Theorem 2.10]).

Lemma 3.1. *Let I be an ideal of R of dimension one and M a finitely generated R-module. Then $H^i_I(M)$ is I-cofinite for all $i \geq 0$.*

Now, we are ready to state and prove the first main result of this section, which is an application of Lemma 2.2.

Theorem 3.2. *Let R be a Noetherian ring, I an ideal of R and M a finitely generated R-module. Then for every $i \geq 0$ and any $j \geq 0$, the set*

$$\{\mathfrak{p} \in \mathrm{Ass}_R \mathrm{Ext}^j_R(R/I, H^i_I(M)) \ : \ \mathrm{ht}(\mathfrak{p}/I) \leq 1\}$$

is finite.

Proof. Suppose there are i and j such that the set

$$\{\mathfrak{p} \in \mathrm{Ass}_R \mathrm{Ext}^j_R(R/I, H^i_I(M)) \ : \ \mathrm{ht}(\mathfrak{p}/I) \leq 1\}$$

is not finite. We can choose an countable set $\{\mathfrak{p}_k\}_{k\geq 1} \subseteq \mathrm{Ass}_R \mathrm{Ext}^j_R(R/I, H^i_I(M))$ and $\mathrm{ht}(\mathfrak{p}_k/I) = 1$ for all $k \geq 1$. Let T as Lemma 2.2, we have $R \to T$ is a flat extension and

$$\mathrm{Max}(T) = \{\mathfrak{p}_k T\}_{k\geq 1}.$$

By the flat base change theorem (see, [2, Theorem 4.3.2]) we have

$$\mathrm{Ext}^j_R(R/I, H^i_I(M)) \otimes_R T \cong \mathrm{Ext}^j_T(R/I \otimes_R T, H^i_I(M) \otimes_R T) \cong \mathrm{Ext}^j_T(T/IT, H^i_{IT}(M \otimes_R T)).$$

So $\mathfrak{p}_k T \in \mathrm{Ass}_T \mathrm{Ext}^j_T(T/IT, H^i_{IT}(M \otimes_R T))$ for all $k \geq 1$ by [10, Theorem 23.2]. On the other hand we have $\dim T/IT = 1$ so $H^i_{IT}(M \otimes_R T)$ is IT-cofinite by Lemma 3.1. Thus the T-module $\mathrm{Ext}^j_T(T/IT, H^i_{IT}(M \otimes_R T))$ is finitely generated and so the set

$$\mathrm{Ass}_T \mathrm{Ext}^j_T(T/IT, H^i_{IT}(M \otimes_R T))$$

is finite, which is a contradiction. The proof is complete. □

Recall that $\mathrm{Ass}_R H^i_I(M) = \mathrm{Ass}_R \mathrm{Hom}(R/I, H^i_I(M))$ for all $i \geq 0$. So the following result is an immediately consequence of Theorem 3.2.

Corollary 3.3. *Let I be an ideal of R and M a finitely generated R-module. Then for every $i \geq 0$ the set $\{\mathfrak{p} \in \mathrm{Ass}_R H^i_I(M) \ : \ \mathrm{ht}(\mathfrak{p}/I) \leq 1\}$ is finite.*

The following results are other applications of Lemma 2.2 to local cohomology modules.

Corollary 3.4. *Let R be a Noetherian ring, I an ideal of R and $n \geq 1$ be an integer and M be a finitely generated R-module such that $\dim(M/IM) = n$. Then for any finitely generated R-module N with support in $V(I + \mathrm{Ann}_R(M))$ and for any $i, j \geq 0$ we have the set*

$$\{\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Ext}^j_R(N, H^i_I(M))) \ : \ \dim(R/\mathfrak{p}) \geq n - 1\}$$

is finite.

Proof. Let $J = \mathrm{Ann}(M/IM)$. Then, we have $V(J) = V(I + \mathrm{Ann}_R(M))$. It is not difficult to see that $H^i_I(M) \cong H^i_J(M)$ for all $i \geq 0$. We can assume henceforth that $I = \mathrm{Ann}(M/IM)$ and $\dim R/I = n$. Notice that if K is an I-cofinite module, then $\mathrm{Ext}^j_R(N, K)$ is finitely generated for all finitely generated R-module N with support $V(I)$ (see [8, Lemma 1]). Now the proof is the same as Theorem 3.2. □

Corollary 3.5. *Let R be a Noetherian ring, I an ideal of R and $n \geq 1$ be an integer and M be a finitely generated R-module such that $\dim(M/IM) = n$. Then for any finitely generated R-module N with support in $V(I + \mathrm{Ann}_R(M))$ and for any $i, j \geq 0$ we have the set*

$$\{\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Tor}_j^R(N, H_I^i(M))) \ : \ \dim(R/\mathfrak{p}) \geq n - 1\}$$

is finite.

Proof. Use [11, Theorem 2.1]. □

We prove the second main result of this section.

Theorem 3.6. *Let R be a Noetherian ring, I an ideal of R and M an (not necessarily finitely generated) R-module. Then for any integer $t \geq 0$, the set*

$$\mathcal{S} := \{\mathfrak{p} \in \mathrm{Ass}_R H_I^t(M) \ : \ \mathrm{ht}(\mathfrak{p}) = t\} = \{\mathfrak{p} \in \mathrm{Supp}(H_I^t(M)) \ : \ \mathrm{ht}(\mathfrak{p}) = t\}$$

is finite.

Proof. It follows from Grothendieck's Vanishing Theorem, that each element of the set $\{\mathfrak{p} \in \mathrm{Supp}(H_I^t(M)) \ : \ \mathrm{ht}(\mathfrak{p}) = t\}$ is a minimal element of the set $\mathrm{Supp}(H_I^t(M))$ and so is an associated prime ideal of the R-module $H_I^t(M)$. Therefore

$$\{\mathfrak{p} \in \mathrm{Supp}(H_I^t(M)) \ : \ \mathrm{ht}(\mathfrak{p}) = t\} \subseteq \mathcal{S} \subseteq \{\mathfrak{p} \in \mathrm{Supp}(H_I^t(M)) \ : \ \mathrm{ht}(\mathfrak{p}) = t\}.$$

Hence

$$\mathcal{S} = \{\mathfrak{p} \in \mathrm{Supp}(H_I^t(M)) \ : \ \mathrm{ht}(\mathfrak{p}) = t\}.$$

Let $\mathfrak{p}$ be an arbitrary element of $\{\mathfrak{p} \in \mathrm{Supp}(H_I^t(M)) \ : \ \mathrm{ht}(\mathfrak{p}) = t\}$ we have $H_{IR_\mathfrak{p}}^t(M_\mathfrak{p}) \neq 0$. Notice that $\dim R_\mathfrak{p} = t$ so by [2, Exercise 6.1.9] we have $H_{IR_\mathfrak{p}}^t(M_\mathfrak{p}) = H_{IR_\mathfrak{p}}^t(R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} M_\mathfrak{p}$. Hence $H_{IR_\mathfrak{p}}^t(R_\mathfrak{p}) \neq 0$. Thus for any R-module M we have

$$\{\mathfrak{p} \in \mathrm{Supp}(H_I^t(M)) \ : \ \mathrm{ht}(\mathfrak{p}) = t\} \subseteq \{\mathfrak{p} \in \mathrm{Supp}(H_I^t(R)) \ : \ \mathrm{ht}(\mathfrak{p}) = t\}.$$

So it is enough to prove the assertion in the case $M = R$. Suppose that $\{\mathfrak{p} \in \mathrm{Ass}_R H_I^t(R) \ : \ \mathrm{ht}(\mathfrak{p}) = t\}$ is not finite. Then, we can choose a countable infinite subset

$$\{\mathfrak{p}_i\}_{i \geq 1} \subseteq \{\mathfrak{p} \in \mathrm{Ass}_R H_I^t(R) \ : \ \mathrm{ht}(\mathfrak{p}) = t\}.$$

Now set T as in Lemma 2.2. Then we have $R \to T$ is a flat extension and $\mathrm{Max}(T) = \{\mathfrak{p}_i T\}_{i \geq 1}$. In particular, T is a Noetherian ring of dimension t and $\mathfrak{p}_i T \in \mathrm{Ass}_T H_{IT}^t(T)$ for all $i \geq 1$. But, in view of [11, Proposition 5.1], the T-module $H_{IT}^t(T)$ is Artinian and hence has finitely many associated primes, which is a contradiction. The proof is complete.

□

Let T be a subset of $\operatorname{Spec}(R)$. We denote

$$T_i = \{\mathfrak{p} \in T \,:\, \operatorname{ht}(\mathfrak{p}) = i\}.$$

The following is a direct consequence of Theorem 3.6.

Corollary 3.7. *Let M be an R-module of finite dimension and I an ideal of R. Then the set*

$$\bigcup_{i\geq 0} (\operatorname{Ass}_R H_I^i(M))_i$$

is finite.

We close this paper with the following remark.

Remark 3.8. It is not known whether the set of minimal associated primes of a local cohomology module is finite. It is equivalent to the question: Is the support of local cohomology closed (see [6])? By Lemma 2.2 one can assume that the set of minimal associated primes of $H_I^i(M)$ is just $\operatorname{Max}(R)$.

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